CS 170 DIS 08

Released on 2019-03-11

1 A HeLPful Introduction

Find necessary and sufficient conditions on real numbers a and b under which the linear program

$$\begin{aligned} \max \ & x + y \\ ax + by &\leq 1 \\ x, y &\geq 0 \end{aligned}$$

(a) Is infeasible.

(b) Is unbounded.

(c) Has a unique optimal solution.

Solution:

(a) Note that for any possible real valued a and b, the point $x = 0, y = 0$ is always feasible (since $a \cdot 0 + b \cdot 0 = 0 \leq 1$). Thus, the program is *not* infeasible for any choice of a and b.

(b) If the LP is unbounded, then the dual LP cannot be feasible (since, otherwise, by weak duality, the feasible solution for the dual will give an upper bound for the primal objective). Next, if an LP is bounded, then by the duality theorem, its (bounded) optimum is the same as that of its dual, which must therefore be feasible. Thus, this LP is unbounded if and only if its dual is infeasible.

We therefore write the dual of the original LP

$$\begin{aligned} \min z & \\ az &\geq 1 \\ bz &\geq 1 \\ z &\geq 0 \end{aligned}$$

Suppose $a > 0$ and $b > 0$. In this case, $z = \max(1/a, 1/b)$ is a feasible solution for the dual. Thus, at least one of a, b must be non-positive for the dual to be infeasible. Suppose $b \leq 0$. Then we have $bz \leq 0 < 1$ for $z \geq 0$, so that the dual is infeasible. Similarly when $a \leq 0$ the dual is infeasible. We conclude that the dual is infeasible if and only if $a \leq 0$ or $b \leq 0$. Thus, from the discussion above, the original LP is unbounded if and only if $\boxed{\text{at least one of } a \text{ and } b \text{ is non-positive}}$.

(c) From part (b), we see that the LP has a bounded optimum if and only if $a > 0$ and $b > 0$. In this case, the feasible region is bounded by the lines $x = 0$ and $y = 0$ and $ax + by = 1$, so that its vertices are $(0, 0), (1/a, 0)$ and $(0, 1/b)$. Since $a, b > 0$, the optimum is $\max(1/a, 1/b)$, and is attained at one of the last two vertices.

Since every point inside the feasible region of an LP can be written down as a convex combination of vertices, it follows that the optimum is non-unique if and only if it is attained at at least two vertices. Since the only two vertices where the optimum can be attained in our LP are $(0, 1/b)$ and $(1/a, 0)$, we will have a unique optimum as long as the objective values at these points, $1/a$ and $1/b$, are different. We therefore get that the LP has a unique bounded optimum if and only if

$$a > 0, b > 0 \text{ and } a \neq b.$$

Alternate Solution. With respect to the parameters a and b, there are 3 possible choices: $a = 0$, $a > 0$, $a < 0$ and similarly for b. Part (a) still holds from above since the point $(0, 0)$ is always feasible. If either of a or b is exactly zeros, then correspondingly x or y can be increased without any bounds and hence the LP is feasible unbounded. A similar logic can be applied if any of a or b is negative; consider the case when a is negative, we can set $y = 0$ and increase x unboundedly (similarly for b negative).

The only case which remains is when both $a > 0$ and $b > 0$. In this case, we can again contruct the bounded region described above in part (c) and the solution follows.

2 Standard Form LP

Recall that any Linear Program can be reduced to a more constrained *standard form* where all variables are nonnegative, the constraints are given by equations and the objective is that of minimizing a cost function. For each of the subparts, what system of variables, constraints, and objectives would be equivalent to the following:

(a) Max Objective: $\max c^\top x$

(b) Min Max Objective: $\min \max(y_1, y_2)$

(c) Upper Bound on Variable: $x_1 \leq b_1$

(d) Lower Bound on Variable: $x_2 \geq b_2$

(e) Bounded Variable: $b_2 \leq x_3 \leq b_1$

(f) Inequality Constraint: $x_1 + x_2 + x_3 \leq b_3$

(g) Unbounded Variable: $x_4 \in R$

Solution:

(a) $\min -c^\top x$

(b) $\min t\,, \quad x \leq t\,, \quad y \leq t$

(c) $x_1 + s_1 = b_1\,, \quad s_1 \geq 0$

(d) $-x_2 + s_2 = -b_2 \quad s_2 \geq 0$

(e) Break it into two inequalities $x_3 \leq b_1$ and $x_3 \geq b_2$ and use the parts above

(f) $x_1 + x_2 + x_3 + s_1 = b_3$, $\quad s_1 \geq 0$

(g) Replace x_4 by $x^+ - x^-$ along with $x^+ \geq 0$, $\quad x^- \geq 0$

3 Job Assignment

There are I people available to work J jobs. The value of person i working 1 day at job j is a_{ij} for $i = 1, \dots, I$ and $j = 1, \dots, J$. Each job is completed after the sum of the time of all workers spend on it add up to be 1 day, though partial completion still has value (i.e. person i working c portion of a day on job j is worth $a_{ij}c$). The problem is to find an optimal assignment of jobs for each person for one day such that the total value created by everyone working is optimized. No additional value comes from working on a job after it has been completed.

(a) What variables should we optimize over? I.e. in the canonical linear programming definition, what is x?

3.5cm

Solution: An assignment x is a choice of numbers x_{ij} where x_{ij} is the portion of person i's time spent on job j.

(b) What are the constraints we need to consider? Hint: there are three major types.

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Solution: First, no person i can work more than 1 day's worth of time.

$$\sum_{j=1}^{J} x_{ij} \leq 1 \qquad \text{for } i = 1, \dots, I.$$

Second, no job j can be worked past completion:

$$\sum_{i=1}^{I} x_{ij} \leq 1 \qquad \text{for } j = 1, \dots, J.$$

Third, we require positivity.

$$x_{ij} \geq 0 \qquad \text{for } i = 1, \dots, I, j = 1, \dots, J.$$

(c) What is the maximization function we are seeking?

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Solution: By person i working job j for x_{ij}, they contribute value $a_{ij}x_{ij}$. Therefore, the net value is

$$\sum_{i=1,j=1}^{I,J} a_{ij}x_{ij} = A \bullet x.$$