Note: Your TA probably will not cover all the problems. This is totally fine, the discussion worksheets are not designed to be finished in an hour. They are deliberately made long so they can serve as a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

1 LP Basics

Linear Program. A *linear program* is an optimization problem that seeks the optimal assignment for a linear objective over linear constraints. Let $x \in \mathbb{R}^n$ be the set of variables and $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$. The canonical form of a linear program is

$$\begin{aligned} &\text{maximize } c^\top x \\ &\text{subject to } Ax \leq b \\ &\qquad\qquad x \geq 0 \end{aligned}$$

Any linear program can be written in canonical form.

Let's check this is the case:

(i) What if the objective is minimization?

(ii) What if you have a constraint $Ax \geq b$?

(iii) What about $Ax = b$?

(iv) What if the constraint is $x \leq 0$?

(v) What about unconstrained variables $x \in \mathbb{R}$?

2 LP Meets Linear Regression

One of the most important problems in the field of *statistics* is the *linear regression problem.* Roughly speaking, this problem involves fitting a straight line to statistical data represented by points (x_1, y_1), $(x_2, y_2), \ldots, (x_n, y_n)$ on a graph. Denoting the line by $y = a + bx$, the objective is to choose the constants a and b to provide the "best" fit according to some criterion. The criterion usually used is the *method of least squares*, but there are other interesting criteria where linear programming can be used to solve for the optimal values of a and b.

Suppose instead we wish to minimize the sum of the absolute deviations of the data from the line:

$$\min \sum_{i=1}^{n} |y_i - (a + bx_i)|$$

Write a linear program with variables a, b to solve this problem.

Hint: Create new variables z_i and new constraints to help represent $|y_i - (a + bx_i)|$ in a linear program.

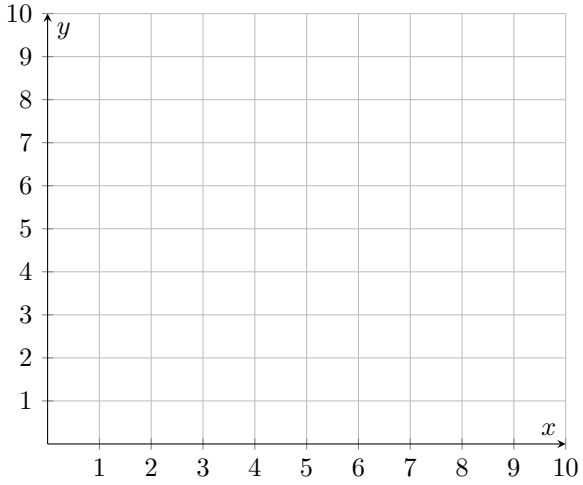
3 Simply Simplex

Consider the following linear program.

$$\max 2x + 3y$$

$$\text{subject to } \begin{cases} x + 3y \geq 6 \\ 2x - y \leq 12 \\ x + y \leq 9 \\ x \geq 0, y \geq 0 \end{cases}$$

(a) Sketch the feasible region below.



(b) Run the Simplex algorithm on this LP starting at $(6, 0)$. What are the vertices visited? **Please show your work.**

4 Huffman and LP

Consider the following Huffman code for characters a, b, c, d: $a = 0, b = 10, c = 110, d = 111$.

Let f_a, f_b, f_c, f_d denote the fraction of characters in a file (only containing these characters) that are a, b, c, d respectively. Write a linear program with variables f_a, f_b, f_c, f_d to solve the following problem: What values of f_a, f_b, f_c, f_d (that can generate this Huffman code) result in the Huffman code using the most bits per character?

5 Job Assignment

There are I people available to work J jobs. The value of person i working 1 day at job j is a_{ij} for $i = 1, \ldots, I$ and $j = 1, \ldots, J$. Each job is completed after the sum of the time of all workers spend on it add up to be 1 day, though partial completion still has value (i.e. person i working c portion of a day on job j is worth $a_{ij}c$). The problem is to find an optimal assignment of jobs for each person for one day such that the total value created by everyone working is optimized. No additional value comes from working on a job after it has been completed.

(a) What variables should we optimize over? In other words, in the canonical linear programming definition, what is x?

(b) What are the constraints we need to consider? Hint: there are three major types.

(c) What is the maximization function we are seeking?

6 Taking a Dual

Consider the following linear program:

$$\begin{aligned} \max\ & 4x_1 + 7x_2 \\ & x_1 + 2x_2 \leq 10 \\ & 3x_1 + x_2 \leq 14 \\ & 2x_1 + 3x_2 \leq 11 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Construct the dual of the above linear program.

7 Egg Drop Revisited

Recall the Egg Drop problem from Homework 7:

You are given m identical eggs and an n story building. You need to figure out the highest floor $b \in \{0, 1, 2, \ldots n\}$ that you can drop an egg from without breaking it. Each egg will never break when dropped from floor b or lower, and always breaks if dropped from floor $b+1$ or higher. ($b = 0$ means the egg always breaks). Once an egg breaks, you cannot use it any more. However, if an egg does not break, you can reuse it.

Let $f(n, m)$ be the minimum number of egg drops that are needed to find b (regardless of the value of b).

Instead of solving for $f(n, m)$ directly, we define a new subproblem $M(x, m)$ to be the maximum number of floors for which we can always find b in at most x drops using m eggs.

For example, $M(2, 2) = 3$ because a 3-story building is the tallest building such that we can always find b in at most 2 egg drops using 2 eggs.

(a) Find a recurrence relation for $M(x, m)$ that can be computed in constant time given the previous subproblems. Briefly justify your recurrence.

Hint: As a starting point, what is the highest floor that we can drop the first egg from and still be guaranteed to solve the problem with the remaining $x - 1$ drops and $m - 1$ eggs if the egg breaks?

(b) Give an algorithm to compute $M(x, m)$ given x and m and analyze its runtime.

(c) Briefly justify the correctness of your algorithm via an inductive proof.

(d) Modify your algorithm from (b) to compute $f(n, m)$ given n and m.
Hint: If we can find b when there are more than n floors, we can also find b when there are n floors.

(e) Show that the runtime of the algorithm from part (c) is $O(nm)$. Compare this to the runtime you found in last week's homework.

(f) Show that we can implement the algorithm from part (c) to use only $O(m)$ space.

