CS 170 DIS 09

Released on 2018-10-22

1 Provably Optimal

Consider the following linear program:

$$\begin{aligned} \max \ & x_1 - 2x_3 \\ & x_1 - x_2 \leq 1 \\ & 2x_2 - x_3 \leq 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

For the linear program above,

(a) First compute the dual of the above linear program

(b) show that the solution $(x_1, x_2, x_3) = (3/2, 1/2, 0)$ is optimal **using its dual**. You do not have to solve for the optimum of the dual. (*Hint:* Recall that any feasible solution of the dual is an upper bound on any feasible solution of the primal)

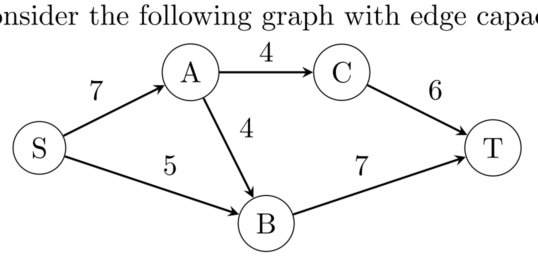
Solution: The dual of the given LP is:

$$\begin{aligned} \min \quad & y_1 + y_2 \\ y_1 & \geq 1 \\ -y_1 + 2y_2 & \geq 0 \\ -y_2 & \geq -2 \\ y_1, y_2 & \geq 0 \end{aligned} \quad \equiv \quad \begin{aligned} \min \quad & y_1 + y_2 \\ y_1 & \geq 1 \\ y_2 & \geq \frac{y_1}{2} \\ y_2 & \leq 2 \\ y_1, y_2 & \geq 0 \end{aligned}$$

The objective value at the claimed optimum is $3/2$. By the duality theorem, this would be optimum if and only if there is a feasible solution to the dual LP with the same objective value. Greedily trying to make y_1, y_2 as small as possible results in finding that$y_1 = 1, y_2 = 1/2$ is a feasible dual solution, with the objective value $3/2$. Thus, the claimed primal optimal is indeed an optimal solution.

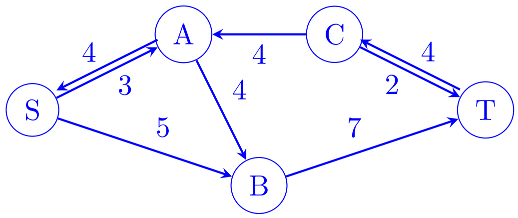
2 Residual in graphs

Consider the following graph with edge capacities as shown:



(a) Consider pushing 4 units of flow through $S \to A \to C \to T$. Draw the residual graph after this push.

Solution:

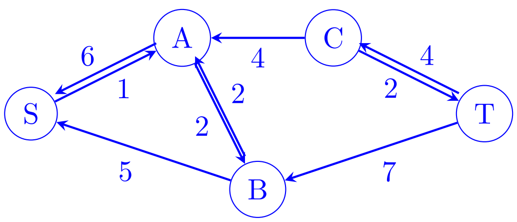


(b) Compute a maximum flow of the above graph. Find a minimum cut. Draw the residual graph of the maximum flow.

Solution: A maximum flow of value 11 results from pushing:

- 4 units of flow through $S \to A \to C \to T$;
- 5 units of flow through $S \to B \to T$; and
- 2 units of flow through $S \to A \to B \to T$.

(There are other maximum flows of the same value, can you find them?) The resulting residual graph (with respect to the maximum flow above) is:



A minimum cut of value 11 is between $\{S, A, B\}$ and $\{C, T\}$ (with cross edges $A \to C$ and $B \to T$).

3 Verifying a max-flow

Suppose someone presents you with a solution to a max-flow problem on some network. Give a *linear* time algorithm to determine whether the solution does indeed give a maximum flow.

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Solution: The max-flow algorithm has found the maximum flow when there is no $s - t$ path in the residual graph. Therefore, we just search for an $s - t$ path in the residual graph of the given flow to see if the given flow is maximal.

procedure CHECKFLOW(G, f)
 Check that $\forall v \in V, v \neq s, t$, $\sum_{(u,v)\in E} f_{uv} = \sum_{(v,w)\in E} f_{vw}$
 Compute G^f, the residual flow network of f.
 Run BFS(G^f, s)
 If BFS finds an s-t path, return false, otherwise return true.

Checking that f is a valid flow takes $O(|V| + |E|)$ time. Constructing G^f takes $O(|V| + |E|)$ time. Running BFS on G^f takes $O(|V| + |E|)$ time since G^f has $|V|$ vertices and $\leq 2|E|$ edges. Therefore, the algorithm is $O(|V| + |E|)$, which is linear.