# CS 170 DIS 11

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## 1 Weighted Rock-Paper-Scissors

You and your friend used to play rock-paper-scissors, and have the loser pay the winner 1 dollar. However, you then learned in CS170 that the best strategy is to pick each move uniformly at random, which took all the fun out of the game.

Your friend, trying to make the game interesting again, suggests playing the following variant: If you win by beating rock with paper, you get 5 dollars from your opponent. If you win by beating scissors with rock, you get 3 dollars. If you win by beating paper with scissors, you get 1 dollar.

(a) Draw the payoff matrix for this game.

(b) Write a linear program to find the optimal strategy.

**Solution:**

(a)

| | | Your Friend: rock | paper | scissors |
|---|---|---|---|---|
| You: | rock | 0 | -5 | 3 |
| | paper | 5 | 0 | -1 |
| | scissors | -3 | 1 | 0 |

(b) Let $r$, $p$, $s$ be the probabilities that you play rock, paper, scissors respectively. Let $z$ stand for the expected payoff, if your opponent plays optimally as well.

$$\begin{aligned}
\max \quad & z \\
5p - 3s &\geq z && \text{(Opponent chooses rock)} \\
s - 5r &\geq z && \text{(Opponent chooses paper)} \\
3r - p &\geq z && \text{(Opponent chooses scissors)} \\
r + p + s &= 1 \\
r, p, s &\geq 0
\end{aligned}$$

## 2 Secret Santa

Imagine you are throwing a party and you want to play Secret Santa. Thus you would like to assign to every person at the party another partier to whom they must anonymously give a gift. However, there are some restrictions on who can give gifts to who. For instance, nobody should be assigned to give a gift to themselves or to their spouse. Since you are the host, you know all of these restrictions. Give an efficient algorithm that determines if you and your guests can play Secret Santa. **Solution:** Let $n$ be the number of guests. For guest $i$, make two vertices $u_i$ and $v_i$. Let $U = \{u_i : i = 1, \dots, n\}$ and $V = \{v_i : i = 1, \dots, n\}$. Construct a graph $G = (U \cup V, E)$, where there is an edge between $u_i$ and $v_j$ if guest $i$ can give a gift to guest $j$. You can play Secret Santa iff $G$ has a perfect matching. Run max-flow on $G$ with edge capacities $c_e = 1\ \forall e \in E$ to get a flow $f$. size($f$) is the size of the largest matching, so $G$ has a perfect matching iff size($f$) $= n$.

## 3 Existence of Perfect Matchings

Prove the following theorem: Let $G = (L \cup R, E)$ be a bipartite graph. Then $G$ has a perfect matching if and only if, for every set $X \subseteq L$, $X$ is connected to at least $|X|$ vertices in $R$. Note that you must prove both directions. (Hint: Use the max-flow-min-cut theorem.)

**Solution:** Assume $G$ has a perfect matching, and consider a subset $X \subseteq L$. Every vertex in $X$ is matched to distinct vertices in $R$, so in particular the neighborhood of $X$ is of size at least $|X|$ since it contains the vertices matched to vertices in $X$.

Assume that every subset $X \subseteq L$ is connected to at least $|X|$ vertices in $R$. Add two vertices $s$ and $t$, and connect $s$ to every vertex in $L$, and $t$ to every vertex in $R$. Let each edge have capacity one. We will lower bound the size of any cut separating $s$ and $t$. Let $C$ be any cut, and let $L = X \cup Y$, where $X$ is on the same side of the cut as $s$, and $Y$ is on the other side. There is an edge from $s$ to each vertex in $Y$, contributing at least $|Y|$ to the value of the cut. Now there are at least $|X|$ vertices in $R$ that are connected to vertices in $X$. Each of these vertices is also connected to $t$, so regardless of which side of the cut they fall on, each vertex contributes one edge cut (either the edge to $t$, or the edge to a vertex in $X$, which is on the same side as $s$). Thus the cut has value at least $|X| + |Y| = |L|$, and by the max-flow min-cut theorem, this implies that the max-flow has value at least $|L|$, which implies that there must be a perfect matching.