Note: Your TA probably will not cover all the problems. This is totally fine, the discussion worksheets are not designed to be finished in an hour. They are deliberately made long so they can serve as a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

1 Breaking Encryption

After years of research, Horizon Wireless released an encryption algorithm E that encrypts an n-bit message x in time $O(n^2)$. Show that if $\mathsf{P} = \mathsf{NP}$ then this encryption algorithm can be broken in polynomial time. More precisely, argue that if $\mathsf{P} = \mathsf{NP}$, then the following decryption problem can be solved in polynomial time.

DECRYPT :

Input: An encrypted message m_e (encrypted using the algorithm E on an unkown input)

Output: Decryption x of the message m_e, i.e an n bit string x such that encrypting x produces m_e.

2 Public Funds

You are looking to build a new fence for your mansion, to keep out pesky people protesting profligate purchases. You have m bank accounts at your disposal to use to pay for your fence; each account i has a balance of b_i. You must choose one of n options for your fence; each fence j costs c_j dollars. You would like to withdraw from at most k of the bank accounts to build the fence, and due to peculiar UC accounting rules, if you use a particular bank account, you must use the whole balance (all b_m dollars.)

Determine whether it is possible to exactly pay for some fence j; that is, whether there is a j between 1 and n such that you can withdraw exactly c_j dollars given the bank account balances $b_1, \ldots, b_m$, the fence costs $c_1, \ldots, c_n$, and k.

Your task is to prove that Public Funds is **NP**-complete.

(a) Prove that Public Funds is in **NP**.

(b) Prove that Public Funds is **NP**-hard by providing a reduction from Subset Sum.

Note: to rigorously prove the correctness of a reduction from A to B, you must show two things:

1. *If an instance of A has a solution, then the transformed instance of B has a solution.*
2. *If an instance of B in the format of the transformation has a solution, then the corresponding instance of A has a solution.*

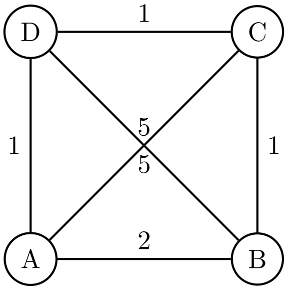
3 Approximating the Traveling Salesperson Problem

Recall in lecture, we learned the following approximation algorithm for TSP:

1. Given the complete graph $G = (V, E)$, compute its MST.
2. Run DFS on the MST computed in the previous step, and record down the the vertices visited in pre-order.
3. The output tour is the list of vertices computed in the previous step, with the first vertex appended to the end.

When G satisfies the triangle inequality, this algorithm achieves an approximation factor of 2. But what happens when triangle inequality does not hold?

Suppose we run this approximation algorithm on the following graph:



The algorithm will return different tours based on the choices it makes during its depth first traversal.

1. Which DFS traversal leads to the best possible output tour?

2. Which DFS traversal leads to the worst possible output tour?

3. What is the approximation ratio given by the algorithm in the worst case for the above instance? Why is it worse than 2?

4 Boba Shops

A rectangular city is divided into a grid of $m \times n$ blocks. You would like to set up boba shops so that for every block in the city, either there is a boba shop within the block or there is one in a neighboring block (assume there are up to 4 neighboring blocks for every block). It costs r_{ij} to rent space for a boba shop in block ij.

Write an integer linear program to determine on which blocks to set up the boba shops, so as to minimize the total rental costs.

(a) What are your variables, and what do they mean?

(b) What is the objective function? Briefly justify.

(c) What are the constraints? Briefly justify.

(d) Solving the non-integer version of the linear program yields a real-valued solution. How would you round the LP solution to obtain an integer solution to the problem? Describe the algorithm in at most two sentences.

(e) What is the approximation ratio of your algorithm in part (d)? Briefly justify.

5 $\sqrt{n}$ coloring

(a) Let G be a graph of maximum degree δ. Show that G is $(\delta + 1)$-colorable.

(b) Suppose $G = (V, E)$ is a 3-colorable graph. Let v be any vertex in G. Show that the graph induced on the neighborhood of v is 2-colorable.

Note: the graph induced on the neighborhood of v refers to the following subgraph:

$$G' = (V' = \text{neighbors of } v, E' = \text{all edges in } E \text{ with both endpoints in } V').$$

(c) Give a polynomial time algorithm that takes in a 3-colorable n-vertex graph G as input and outputs a valid coloring of its vertices using $O(\sqrt{n})$ colors. Prove that your algorithm is correct.

Hint: think of an algorithm that first assigns colors to "high-degree" vertices and their neighborhoods, and then assigns colors to the rest of the graph. The previous two parts might be useful.