CS 170 HW 8

Due 2019-10-23, at 10:00 pm

1 Study Group

List the names and SIDs of the members in your study group.

2 Three Partition

Given a list of positive numbers, \(a_1, \ldots, a_n\), determine if we can partition \(\{1, \ldots, n\}\) into 3 disjoint subsets, \(I, J, K\) such that:

\[
\sum_{i \in I} a_i = \sum_{j \in J} a_j = \sum_{k \in K} a_k = \frac{\sum_{i=1}^{n} a_i}{3}
\]

Devise and analyze a dynamic programming solution to the above problem that runs in time polynomial in \(\sum_{i=1}^{n} a_i\) and \(n\).

Please provide a 3-part solution.

3 Maximum weight independent set

Let \(G = (V, E)\) be a weighted graph, with nonnegative weights \(w(v)\) for each vertex \(v \in V\). A subset of nodes \(S \subset V\) is an independent set of \(G\) if there are no edges between them. Assuming that \(G\) is a tree, find a linear time algorithm for finding the maximum weight independent set in \(G\), i.e. an independent set \(S\) of \(G\) such that \(\sum_{v \in S} w(v)\) is maximized.

Please provide a 3-part solution.

4 (Linear Programming) Minimum Spanning Trees

Consider the minimum spanning tree problem, where we are given an undirected graph \(G\) with edge weights \(w_{u,v}\) for every pair of vertices \(u, v\).

An integer linear program that solves the minimum spanning tree problem is as follows:

Minimize \(\sum_{(u,v) \in E} w_{u,v}x_{u,v}\)

subject to

\[
\sum_{\{u,v\} \in E: u \in S, v \in V \setminus S} x_{u,v} \geq 1 \quad \text{for all } S \subseteq V \text{ with } 0 < |S| < |V|\
\]

\[
\sum_{\{u,v\} \in E} x_{u,v} \leq |V| - 1
\]

\(x_{u,v} \in \{0, 1\}, \quad \forall (u,v) \in E\)
(a) Show how to obtain a minimum spanning tree $T$ of $G$ from an optimal solution of the ILP, and prove that $T$ is indeed an MST. Why do we need the constraint $x_{u,v} \in \{0, 1\}$?

(b) How many constraints does the program have?

(c) Suppose that we replaced the binary constraint on each of the decision variables $x_{u,v}$ with the pair of constraints:

$$0 \leq x_{u,v} \leq 1, \quad \forall (u, v) \in E$$

How does this affect the optimal value of the program? Give an example of a graph where the optimal value of the relaxed linear program differs from the optimal value of the integer linear program.

5 Jeweler

You are a jeweler who sells necklaces and rings. Each necklace takes 4 ounces of gold and 2 diamonds to produce, each ring takes 1 ounce of gold and 3 diamonds to produce. You have 80 ounces of gold and 90 diamonds. You make a profit of 60 dollars per necklace you sell and 30 dollars per ring you sell, and want to figure out how many necklaces and rings to produce to maximize your profits.

(a) Formulate this problem as a linear programming problem and find the solution (state the cost function, linear constraints, and all vertices except for the origin).

(b) Suppose instead that the profit per necklace is $C$ dollars and the profit per ring remains at 30 dollars. For each vertex you listed in the previous part, give the range of $C$ values for which that vertex is the optimal solution.

6 Modeling: Tricks of the Trade

One of the most important problems in the field of statistics is the linear regression problem. Roughly speaking, this problem involves fitting a straight line to statistical data represented by points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ on a graph. Denoting the line by $y = a + bx$, the objective is to choose the constants $a$ and $b$ to provide the “best” fit according to some criterion. The criterion usually used is the method of least squares, but there are other interesting criteria where linear programming can be used to solve for the optimal values of $a$ and $b$. For each of the following criteria, formulate the linear programming model for this problem.

1. Minimize the sum of the absolute deviations of the data from the line; that is,

$$\min \sum_{i=1}^{n} |y_i - (a + bx_i)|$$

2. Minimize the maximum absolute deviation of the data from the line; that is,

$$\min \max_{i=1 \ldots n} |y_i - (a + bx_i)|$$