CS 170 HW 9

Due 2019-10-30, at 10:00 pm

1 Study Group

List the names and SIDs of the members in your study group.

2 Vacation

You are given a connected undirected graph $G = (V, E)$. Recall that a set of vertices $S \subseteq V$ is an independent set if there do not exist $u, v \in S$ such that there is an edge between $u$ and $v$. In addition, an edge cover is a set of edges $C \subseteq E$ such that for each vertex $v$, there is some edge in $C$ that it is incident to (so the edges in $C$ ‘cover’ all the vertices).

(a) In the maximum independent set problem, you want to find an independent set of maximum size. Write the LP for the relaxed version of maximum independent set.

(b) Find the dual LP of the LP you made in part (a). What problem does the dual represent?

(c) True or false: For any connected graph, the optimum value for a) always equals the optimum value for b). If true, prove. If false, give a counterexample.

(d) Consider the ‘non-relaxed’/‘tightened’ version of your LPs for part a) and part b). You can build the tightened LP by adding the constraint that all of the variables also must be integers.

True or false: For any connected graph, the optimum value for the tightened version of a) always equals the optimum value for the tightened version of b). If true, prove. If false, give a counterexample.

3 Generalized Max Flow

Consider the following generalization of the maximum flow problem. You are given a directed network $G = (V, E)$ where edge $e$ has capacity $c_e$. Instead of a single $(s, t)$ pair, you are given multiple pairs $(s_1, t_1), ..., (s_k, t_k)$, where the $s_i$ are sources of $G$ and $t_i$ are sinks of $G$. You are also given $k$ (positive) demands $d_1, ..., d_k$. The goal is to find $k$ flows $f^{(1)}, ..., f^{(k)}$ with the following properties:

(a) $f^{(i)}$ is a valid flow from $s_i$ to $t_i$.

(b) For each edge $e$, the total flow $f^{(1)}_e + f^{(2)}_e + ... + f^{(k)}_e$ does not exceed the capacity $c_e$.

(c) The size of each flow $f^{(i)}$ is at least the demand $d_i$.

(d) The size of the total flow (the sum of the flows) is as large as possible.
Write a linear problem using the variables $f_e^{(i)}$ whose optimal solution is exactly the solution to this problem. For each constraint as well as the objective in your linear program briefly explain why it is correct. (Note: Since linear programs can be solved in polynomial time, this implies a polynomial-time algorithm for the problem)

4 Repairing a Flow

In a particular network $G = (V,E)$ whose edges have integer capacities $c_e$, we have already found a maximum flow $f$ from node $s$ to node $t$ where $f_e$ is an integer for every edge. However, we now find out that one of the capacity values we used was wrong: for edge $(u,v)$ we used $c_{uv}$ whereas it should have been $c_{uv} - 1$. This is unfortunate because the flow $f$ uses that particular edge at full capacity: $f_{uv} = c_{uv}$. We could redo the flow computation from scratch, but there’s a faster way.

Describe an algorithm to repair the max-flow in $O(|V| + |E|)$ time. Also give a proof of correctness and runtime justification.

5 Reductions Among Flows

Show how to reduce the following variants of Max-Flow to the regular Max-Flow problem, i.e. do the following steps for each variant: Given a directed graph $G$ and the additional variant constraints, show how to construct a directed graph $G'$ such that

1. If $F$ is a flow in $G$ satisfying the additional constraints, there is a flow $F'$ in $G'$ of the same size,

2. If $F'$ is a flow in $G'$, then there is a flow $F$ in $G$ satisfying the additional constraints with the same size.

Prove that properties (1) and (2) hold for your graph $G'$.

(a) Max-Flow with Vertex Capacities: In addition to edge capacities, every vertex $v \in G$ has a capacity $c_v$, and the flow must satisfy $\forall v : \sum_{(u,v) \in E} f_{uv} \leq c_v$.

(b) Max-Flow with Multiple Sources: There are multiple source nodes $s_1, \ldots, s_k$, and the goal is to maximize the total flow coming out of all of these sources.

6 A Flowy Metric

Consider an undirected graph $G$ with capacities $c_e \geq 0$ on all edges. $G$ has the property that any cut in $G$ has capacity at least 1. For example, a graph with a capacity of 1 on all edges is connected if and only if all cuts have capacity at least 1. However, $c_e$ can be an arbitrary nonnegative number in general.

1. Show that for any two vertices $s, t \in G$, the max flow from $s$ to $t$ is at least 1.
2. Define the length of a flow \( f \) to be \( \text{length}(f) = \sum_{e \in G} |f_e| \). Define the flow distance \( d_{\text{flow}}(s, t) \) to be the minimum length of any \( s-t \) flow \( f \) that sends one unit of flow from \( s \) to \( t \) and satisfies all capacities; i.e. \( |f_e| \leq c_e \) for all edges \( e \).

Show that if \( c_e = 1 \) for all edges \( e \) in \( G \), then \( d_{\text{flow}}(s, t) \) is the length of the shortest path in \( G \) from \( s \) to \( t \).

(Hint: Let \( d(s, t) \) be the length of the shortest path from \( s \) to \( t \). A good place to start might be to first try to show \( d_{\text{flow}}(s, t) \leq d(s, t) \). Then try to show \( d_{\text{flow}}(s, t) \geq d(s, t) \))

3. (Optional) The shortest path satisfies the triangle inequality, that is for three vertices, \( s, t, \) and \( u \) in \( G \), if \( d(x, y) \) is the length of the shortest path from \( x \) to \( y \), then \( d(s, t) \leq d(s, u) + d(u, t) \). Show that the triangle inequality also holds for the flow distance. That is; show that for any three vertices \( s, t, u \in G \)

\[
d_{\text{flow}}(s, t) \leq d_{\text{flow}}(s, u) + d_{\text{flow}}(u, t)
\]

even when the capacities are arbitrary nonnegative numbers.