CS 170 Homework 10

Due **Saturday 4/12/2025, at 10:00 pm (grace period until 11:59pm)**

1 Study Group (1 point)

List the names and SIDs of the members in your study group. If you have no collaborators, you must explicitly write "none".

2 Applications of Max-Flow Min-Cut (12 points)

Review the statement of max-flow min-cut theorem and prove the following two statements.

(a) (6 points) Let $G = (L \cup R, E)$ be a unweighted bipartite graph[1]. Then G has a L-perfect matching (a matching[2] with size $|L|$) if and only if, for every set $S \subseteq L$, S is connected to at least $|S|$ vertices in R. You must prove both directions.

Hint: Use the max-flow min-cut theorem on the cut that forms S and $L \setminus S$.

(b) (6 points) Let G be an unweighted directed graph and $s, t \in V$ be two distinct vertices. Then the maximum number of edge-disjoint s-t paths equals the minimum number of edges whose removal disconnects t from s (*i.e.*, no directed path from s to t after the removal).

Hint: show how to decompose a flow of value k into k disjoint paths, and how to transform any set of k edge-disjoint paths into a flow of value k.

[1] A bipartite graph $G = (L \cup R, E)$ is defined as a graph that can be partitioned into two disjoint sets of vertices (i.e. L and R) such that no two vertices within the same set are adjacent.

[2] A matching is defined as a set of edges that share no common vertices.

3 Max-Flow Duality (20 points)

In lecture, we discussed how the max-flow problem can be described as an LP and also discussed its relation to the min-cut problem. In this problem, we will connect the max-flow and min-cut problems through LPs directly. For the duration of this problem, we will consider a weighted directed graph $G = (V, E)$ with $s, t \in V$ and capacity weights $w : E \to \mathbb{Z}^{>0}$, and we use the following LP formulation of max-flow (that is in canonical form):

$$\max \sum_{\substack{v \in V \\ (s,v) \in E}} f_{(s,v)}$$

$$\text{such that:} \quad \sum_{\substack{u' \in V \\ (v_0,u') \in E}} f_{(v_0,u')} - \sum_{\substack{u \in V \\ (u,v_0) \in E}} f_{(u,v_0)} \leq 0 \qquad \forall v_0 \in V \setminus \{s,t\}$$

$$\sum_{\substack{u \in V \\ (u,v_0) \in E}} f_{(u,v_0)} - \sum_{\substack{u' \in V \\ (v_0,u') \in E}} f_{(v_0,u')} \leq 0 \qquad \forall v_0 \in V \setminus \{s,t\}$$

$$f_{(u,v)} \leq w_{(u,v)} \qquad \forall (u,v) \in E$$

$$f_{(u,v)} \geq 0 \qquad \forall (u,v) \in E$$

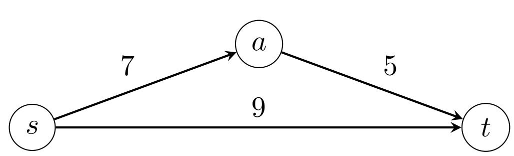
(a) (2 points) Briefly explain the optimization function and each constraint.

(b) (3 points) Take the dual of this LP. *Hint: you will have many equation types. This is expected!*

(c) (2 points) Call the dual LP L. Note that each vertex has potentially multiple variables corresponding to them. Explain how we can group some of the variables to change the dual into a non-canonical form LP, L', that involves one of the variable types from HW8 Q4. *Hint: L' should be slightly cleaner.*

(d) (3 points) Suppose that (for this part only) we enforced the constraint that each of our variables in L' are either 0 or 1. Give an interpretation for each of the variables of L'. *Hint: start with the vertex variables. Make sure to be careful with + and −!*

(e) (3 points) Explain why every optimal solution to the L' LP for the following graph assigns edge variables to either 0 or 1. *Hint: you may have to consider the edge of weight 9 separately.*



(f) (4 points) Prove that, for every arbitrary directed graph (with assigned nodes s and t), there exists an optimal solution to its L' LP assigning all edge variables to either 0 or 1. *Hint: work with edges close to s first.*

(g) (3 points) Explain how the dual and its optimal solution relate to the min-cut problem.

4 Domination (7 points)

In this problem, we explore a concept called *dominated strategies*. Consider a zero-sum game with the following payoff matrix for the row player:

		Column: A	B	C
Row:	D	1	2	-3
	E	3	2	-2
	F	-1	-2	2

Note: All parts of this problem can and should be solved without using an LP solver or solving a system of linear equations.

(a) (3 points) If the row player plays optimally, can you find the probability that they pick D without directly solving for the optimal strategy? Justify your answer.

(b) (2 points) Given the answer to part a, if the both players play optimally, what is the probability that the column player picks A? Justify your answer.

(c) (2 points) Given the answers to part a and b, what are both players' optimal strategies?

5 Weighted Rock-Paper-Scissors (11 points)

For this problem only, you are allowed to use code or online tools to automatically solve your LPs. Feel free to use an LP solver such as: `https://online-optimizer.appspot.com/`. However, **make sure to cite your LP solver** if you use one.

You and your friend used to play rock-paper-scissors, and have the loser pay the winner 1 dollar. However, you then learned in CS 170 that the best strategy is to pick each move uniformly at random, which took all the fun out of the game.

Your friend, trying to make the game interesting again, suggests playing the following variant: If you win by beating rock with paper, you get 2 dollars from your opponent. If you win by beating scissors with rock, you get 1 dollars. If you win by beating paper with scissors, you get 4 dollar.

(a) (1 point) Draw the payoff matrix for this game. Assume that you are the maximizer and the row player, and your friend is the minimizer.

(b) (2 points) Write an LP to find the optimal strategy in your perspective. You do not need to solve the LP.

The following subparts are independent of the previous subparts.

Your friend now wants to make the game even more interesting and suggests that you assign points based on the following payoff matrix:

		Your friend: rock	paper	scissors
You:	rock	-10	3	3
	paper	4	-1	-3
	scissors	6	-9	2

(c) (4 points) Write an LP to find the optimal strategy for yourself. What is the optimal strategy and expected payoff?

Feel free to use an online LP solver.

(d) (4 points) Now do the same for your friend. What is the optimal strategy and expected payoff? How does the expected payoff compare to the answer you get in part (c)?