CS 170 Homework 11

Due **Friday 11/15/2024, at 10:00 pm (grace period until 11:59pm)**

1 Study Group

List the names and SIDs of the members in your study group. If you have no collaborators, explicitly write "none".

2 Decision vs. Search vs. Optimization

Recall that a vertex cover is a set of vertices in a graph such that every edge is adjacent to at least one vertex in this set.

The following are three formulations of the VERTEX COVER problem:

- As a *decision problem*: Given a graph G, return TRUE if it has a vertex cover of size at most b, and FALSE otherwise.
- As a *search problem*: Given a graph G, find a vertex cover of size at most b (that is, return the actual vertices), or report that none exists.
- As an *optimization problem*: Given a graph G, find a minimum vertex cover.

At first glance, it may seem that search should be harder than decision, and that optimization should be even harder. We will show that if any one can be solved in polynomial time, so can the others.

(a) Suppose you are handed a black box that solves VERTEX COVER (DECISION) in polynomial time. Give an algorithm that solves VERTEX COVER (SEARCH) in polynomial time.

(b) Similarly, suppose we know how to solve VERTEX COVER (SEARCH) in polynomial time. Give an algorithm that solves VERTEX COVER (OPTIMIZATION) in polynomial time.

3 Some Sums

Given an array $A = [a_1, a_2, \ldots, a_n]$ of nonnegative integers, consider the following problems:

1 **Partition**: Determine whether there is a subset $S \subseteq [n]$ ($[n] := \{1, 2, \cdots, n\}$) such that $\sum_{i \in S} a_i = \sum_{j \in ([n] \setminus S)} a_j$. In other words, determine whether there is a way to partition A into two disjoint subsets such that the sum of the elements in each subset equal.

2 **Subset Sum**: Given some integer x, determine whether there is a subset $S \subseteq [n]$ such that $\sum_{i \in S} a_i = x$. In other words, determine whether there is a subset of A such that the sum of its elements is x.

3 **Knapsack**: Given some set of items each with weight w_i and value v_i, and fixed numbers W and V, determine whether there is some subset $S \subseteq [n]$ such that $\sum_{i \in S} w_i \leq W$ and $\sum_{i \in S} v_i \geq V$.

For each of the following clearly describe your reduction and justify its correctness.

(a) Find a linear time reduction from Subset Sum to Partition.

(b) Find a linear time reduction from Subset Sum to Knapsack.

4 Reduction to 3-Coloring

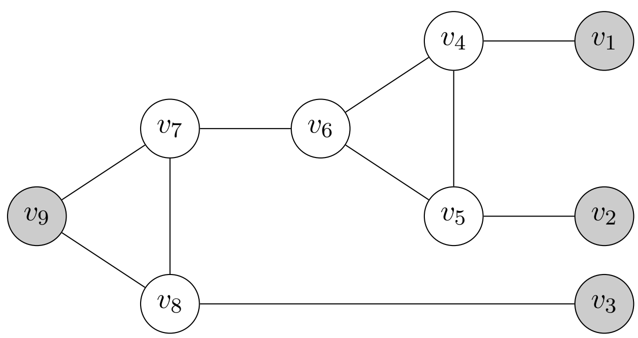
Given a graph $G = (V, E)$, a valid 3-coloring assigns each vertex in the graph a color from {red, green, blue} such that for any edge (u, v), u and v have different colors. In the 3-coloring problem, our goal is to find a valid 3-coloring if one exists. In this problem, we will give a reduction from 3-SAT to the 3-coloring problem. Since we know that 3-SAT is NP-Hard (there is a reduction to 3-SAT from every NP problem), this will show that 3-coloring is NP-Hard (there is a reduction to 3-coloring from every NP problem).

In our reduction, the graph will start with three special vertices, labelled v_{TRUE}, v_{FALSE}, and v_{BASE}, as well as the edges $(v_{\mathsf{TRUE}}, v_{\mathsf{FALSE}})$, $(v_{\mathsf{TRUE}}, v_{\mathsf{BASE}})$, and $(v_{\mathsf{FALSE}}, v_{\mathsf{BASE}})$.

(a) For each variable x_i in a 3-SAT formula, we will create a pair of vertices labeled x_i and $\neg x_i$. How should we add edges to the graph such that in any valid 3-coloring, one of $x_i, \neg x_i$ is assigned the same color as v_{TRUE} and the other is assigned the same color as v_{FALSE}?

Hint: any vertex adjacent to v_{BASE} must have the same color as either v_{TRUE} or v_{FALSE}. Why is this?

(b) Consider the following graph, which we will call a "gadget":



Consider any valid 3-coloring of this graph that does *not* assign the color red to any of the gray vertices (v_1, v_2, v_3, v_9). Show that if v_9 is assigned the color blue, then at least one of $\{v_1, v_2, v_3\}$ is assigned the color blue.

Hint: it's easier to prove the contrapositive!

(c) We have now observed the following about the graph we are creating in the reduction:

(i) For any vertex, if we have the edges (u, v_{FALSE}) and (u, v_{BASE}) in the graph, then in any valid 3-coloring u will be assigned the same color as v_{TRUE}.

(ii) Through brute force one can also show that in a gadget, if all the following hold:

(1) All gray vertices are assigned the color green or blue.

(2) v_9 is assigned the color blue.

(3) At least one of $\{v_1, v_2, v_3\}$ is assigned the color blue.

Then there is a valid coloring for the white vertices in the gadget.

Using these observations and your answers to the previous parts, **give a reduction from 3-SAT to 3-coloring. Prove that your reduction is correct (you do not need to prove any of the observations above).**

Hint: create a new gadget per clause!