CS 170 Homework 12

Due **Saturday 4/26/2025, at 10:00 pm (grace period until 11:59pm)**

1 Study Group (1 point)

List the names and SIDs of the members in your study group. If you have no collaborators, you must explicitly write "none".

2 Reduction to 3-Coloring (12 points)

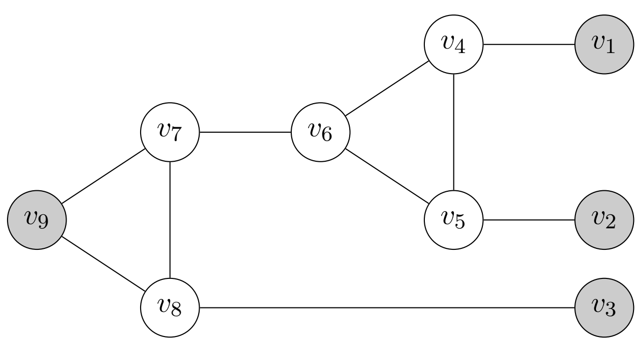
Given a graph $G = (V, E)$, a valid 3-coloring assigns each vertex in the graph a color from $\{$red, green, blue$\}$ such that for any edge (u, v), u and v have different colors. In the 3-coloring problem, our goal is to find a valid 3-coloring if one exists. In this problem, we will give a reduction from 3-SAT to the 3-coloring problem. Since we know that 3-SAT is NP-Hard (there is a reduction to 3-SAT from every NP problem), this will show that 3-coloring is NP-Hard (there is a reduction to 3-coloring from every NP problem).

In our reduction, the graph will start with three special vertices, labelled v_{TRUE}, v_{FALSE}, and v_{BASE}, as well as the edges $(v_{\mathsf{TRUE}}, v_{\mathsf{FALSE}})$, $(v_{\mathsf{TRUE}}, v_{\mathsf{BASE}})$, and $(v_{\mathsf{FALSE}}, v_{\mathsf{BASE}})$.

(a) (3 points) For each variable x_i in a 3-SAT formula, we will create a pair of vertices labeled x_i and $\neg x_i$. How should we add edges to the graph such that in any valid 3-coloring, one of $x_i, \neg x_i$ is assigned the same color as v_{TRUE} and the other is assigned the same color as v_{FALSE}?

Hint: any vertex adjacent to v_{BASE} must have the same color as either v_{TRUE} or v_{FALSE}. Why is this?

(b) (3 points) Consider the following graph, which we will call a "gadget":



Consider any valid 3-coloring of this graph that does *not* assign the color red to any of the gray vertices (v_1, v_2, v_3, v_9). Show that if v_9 is assigned the color blue, then at least one of $\{v_1, v_2, v_3\}$ is assigned the color blue.

Hint: it's easier to prove the contrapositive!

(c) (6 points) We have now observed the following about the graph we are creating in the reduction:

(i) For any vertex, if we have the edges (u, v_{FALSE}) and (u, v_{BASE}) in the graph, then in any valid 3-coloring u will be assigned the same color as v_{TRUE}.

(ii) Through brute force one can also show that in a gadget, if all the following hold:

(1) All gray vertices are assigned the color green or blue.

(2) v_9 is assigned the color blue.

(3) At least one of $\{v_1, v_2, v_3\}$ is assigned the color blue.

Then there is a valid coloring for the white vertices in the gadget.

Using these observations and your answers to the previous parts, **give a reduction from 3-SAT to 3-coloring. Prove that your reduction is correct (you do not need to prove any of the observations above).**

Hint: create a new gadget per clause!

3 Dominating Set (8 points)

A dominating set of a graph $G = (V, E)$ is a subset S of V, such that every vertex not in S is a neighbor of at least one vertex in S. Let the Minimum Dominating Set problem be the task of determining whether there is a dominating set of size $\leq k$. Show that the Minimum Dominating Set problem is NP-Complete. You may assume that G is connected.

Hint: Try reducing from Vertex Cover or Set Cover.

4 k-XOR (12 points)

In the k-XOR problem, we are given n boolean variables $x_1, x_2, \ldots, x_n$, a threshold integer $t > 0$, and a list of m clauses each of which is the XOR of exactly k distinct literals (that is, the clause is true if and only if an odd number of the k literals in the clause are true). Some examples of k-XOR clauses for $k = 3$ are $(x_1 \oplus x_2 \oplus x_3)$ and $(\neg x_1 \oplus x_4 \oplus \neg x_5)$. Our goal is to decide if there is some assignment of variables that satisfies at least t clauses.

(a) (4 points) In the Max-Cut problem, we are given an undirected unweighted graph $G = (V, E)$ and integer α and want to find a cut $S \subseteq V$ such that at least α edges cross this cut (i.e. have exactly one endpoint in S). Give and argue correctness of a reduction from Max-Cut to 2-XOR.

Hint: every clause in 2-XOR is equivalent to an edge in Max-Cut.

(b) (8 points) Give and argue correctness of a reduction from 3-XOR to 4-XOR.

5 Approximating Independent Set (9 points)

In the maximum independent set (MIS) problem, we are given a graph $G = (V, E)$, and our goal is to find the largest set of vertices such that no two vertices in the set are connected by an edge. For this problem, we will assume the degree of all vertices in G is bounded by d (i.e. $\forall v \in V,\ \deg(v) \leq d$).

(a) (2 points) Call a set $S \subseteq V$ a *maximal independent set* if its vertices are pairwise non-adjacent and adding any new vertex to S would make the set no longer an independent set (i.e. no longer satisfy the pairwise non-adjacency property). Give an efficient greedy algorithm for creating a maximal independent set.

(b) (2 points) Consider the following algorithm:

procedure IndSetApprox(V, E)
  Create some maximal independent set I
  return I

Provide an example where the above algorithm does not give an optimal solution.

(c) (5 points) Provide an approximation ratio for the algorithm from part (b) in terms of $|V|$, $|E|$, and/or d. Briefly justify your answer.

6 $\sqrt{n}$ coloring (12 points)

(a) (3 points) Let G be a graph of maximum degree δ. Show that G is $(\delta + 1)$-colorable.

(b) (2 points) Suppose $G = (V, E)$ is a 3-colorable graph. Let v be any vertex in G. Show that the graph induced on the neighborhood of v is 2-colorable.

Note: the graph induced on the neighborhood of v refers to the following subgraph:

$$G' = (V' = \text{neighbors of } v, E' = \text{all edges in } E \text{ with both endpoints in } V').$$

(c) (7 points) Give a polynomial time algorithm that takes in a 3-colorable n-vertex graph G as input and outputs a valid coloring of its vertices using $O(\sqrt{n})$ colors. Prove that your algorithm is correct.

Hint: think of an algorithm that first assigns colors to "high-degree" vertices and their neighborhoods, and then assigns colors to the rest of the graph. The previous two parts might be useful.