1 Study Group

List the names and SIDs of the members in your study group.

2 Dominating Set

A dominating set of a graph $G = (V, E)$ is a subset $D$ of $V$, such that every vertex not in $D$ is a neighbor of at least one vertex in $D$. Let the Minimum Dominating Set problem be the task of determining whether there is a dominating set of size $\leq k$. Show that the Minimum Dominating Set problem is NP-hard. You may assume that $G$ is connected.

3 More Reductions

Given an array $A = [a_1, a_2, \ldots, a_n]$ of nonnegative integers, consider the following problems:

1 **Partition**: Determine whether there is a subset $P \subseteq [n]$ ($[n] := \{1, 2, \ldots, n\}$) such that $\sum_{i \in P} a_i = \sum_{j \in [n] \setminus P} a_j$.

2 **Subset Sum**: Given some integer $k$, determine whether there is a subset $P \subseteq [n]$ such that $\sum_{i \in P} a_i = k$.

3 **Knapsack**: Given some set of items each with weight $w_i$ and value $v_i$, and fixed numbers $W$ and $V$, determine whether there is some subset $P \subseteq [n]$ such that $\sum_{i \in P} w_i \leq W$ and $\sum_{i \in P} v_i \geq V$.

For each of the following clearly describe your reduction, justify runtime and correctness.

(a) Find a linear time reduction from **Subset Sum** to **Partition**.

(b) Find a linear time reduction from **Subset Sum** to **Knapsack**.

4 Randomization for Approximation

Oftentimes, extremely simple randomized algorithms can achieve reasonably good approximation factors.

(a) Consider Max 3-SAT (given a set of 3-clauses, find the assignment that satisfies as many of them as possible). Come up with a simple randomized algorithm that will achieve an approximation factor of $\frac{7}{8}$ in expectation. That is, if the optimal solution satisfies $k$ clauses, your algorithm should produce an assignment that satisfies at least $\frac{7}{8} \ast k$ clauses in expectation. You may assume that every clause contains exactly 3 distinct variables.

(b) Given an instance of Max 3-SAT with $n$ clauses, what is the maximum number of clauses that are guaranteed to be solved in at least one assignment of variables?
(c) Give an example of a Max 3-SAT instance where the optimal solution matches the number in (b).

5 Reduction to 3-Coloring

Given a graph $G = (V, E)$, a valid 3-coloring assigns each vertex in the graph a color from \{0, 1, 2\} such that for any edge $(u, v)$, $u$ and $v$ have different colors. In the 3-coloring problem, our goal is to find a valid 3-coloring if one exists. In this problem, we will give a reduction from 3-SAT to the 3-coloring problem, showing that 3-coloring is NP-hard.

In our reduction, the graph will start with three special vertices, labelled “True”, “False”, and “Base”, and the edges (True, False), (True, Base), and (False, Base).

(a) For each variable $x_i$ in a 3-SAT formula, we will create a pair of vertices labeled $x_i$ and $\neg x_i$. How should we add edges to the graph such that in any valid 3-coloring, one of $x_i$, $\neg x_i$ is assigned the same color as True and the other is assigned the same color as False?

(b) Consider the following graph, which we will call a “gadget”:

Show that in any valid 3-coloring of this graph which does not assign the color 2 to any of the gray vertices, the gray vertex on the right is assigned the color 1 only if one of the gray vertices on the left is assigned the color 1.

(c) We observe the following about the graph we are creating in the reduction:

(i) For any vertex, if we have the edges $(v, \text{False})$ and $(v, \text{Base})$ in the graph, then in any valid 3-coloring $v$ will be assigned the same color as True.

(ii) Through brute force one can also show that in the gadget, for any assignment of colors to gray vertices such that:

(1) All gray vertices are assigned the color 0 or 1
(2) The gray vertex on the right is assigned the color 1
(3) At least one gray vertex on the left is assigned the color 1

Then there is a valid coloring for the black vertices in the gadget.

Using these observations and your answers to the previous parts, give a reduction from 3-SAT to 3-coloring. Prove that your reduction is correct.
6 Project: Drive the TAs Home

1. Logistics: Read the project spec to answer the following questions.
   (a) When are the Phase 1 and Phase 2 due dates?
   (b) What are the two deliverables in Phase 1?
   (c) What are the three deliverables in Phase 2?
   (d) Under what circumstances are you allowed to make changes to your group after 11/13?
   (e) How many days are you allowed to submit after the deadline?

2. Cost Function
   (a) Assume Rao and 2 TAs are in the car, and the car moves a distance of 5 (no TAs get dropped off in this process). How much cost does this incur?
   (b) What is the cost of the example output in the spec? Graph of input is given below. All edge weights are 1.

   1. Dwinelle
   2. Wheeler
   3. Campanile
   0. Soda
   4. Cory
   5. RSF
   6. Barrows

   (c) Give the output with the optimal cost.

3. In the metric TSP problem, we are given a graph $G = (V, E)$ with non-negative edge weights that satisfies the triangle inequality, and we want to find a minimum weight tour visiting each vertex at least once.
   (a) Show that if the driver takes $\frac{1}{2}$ the energy of a TA to travel the same unit of distance as walking, it would always be optimal to drop every TA off at their own home.
   (b) Give a reduction from metric TSP to Drive the TAs Home where it takes the driver $\frac{1}{2}$ the energy of a TA to travel the same unit of distance. Assume here that metric TSP can visit each vertex more than once.