Lecture 14

Linear Programming

LINEAR PROGRAMMING EXAMPLE:

BAKERY:

INGREDIENTS NEEDED TO MAKE

	Flour	Sugar	Eggs
Donuts	2	2	7
Cake	5	9	12
INGREDIENTS AVAILABLE:	200	300	500

PROFIT PER DONUTS = 5 PER CAKE = 25

Decision Variables:

$x \rightarrow$ # of donuts

$y \rightarrow$ # of cakes

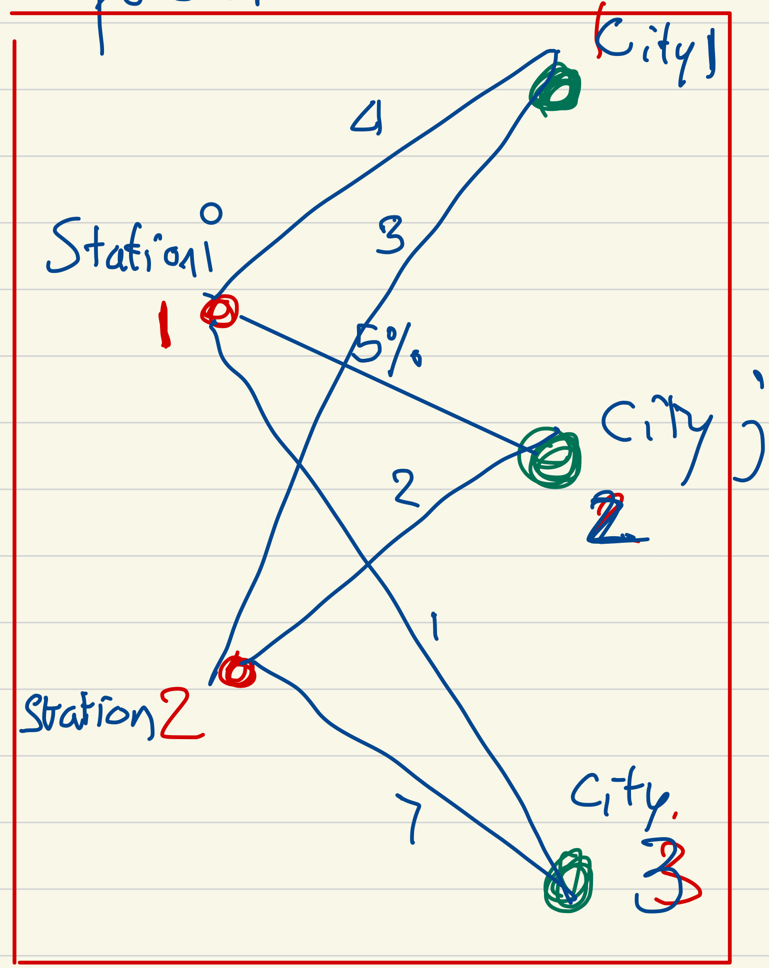
$$\boxed{\begin{aligned} x, y &\geq 0 \\ 2x + 5y &\leq 200 \\ 2x + 9y &\leq 300 \\ 7x + 12y &\leq 500 \end{aligned}}$$

Maximise $5x + 25 \cdot y$

LINEAR PROGRAMMING EXAMPLE 2

POWER STATIONS

* 2 power stations that deliver power to 3 cities



* DEMAND:

City	Demand
1	40
2	60
3	80

* Each unit of power from Station i to City j incurs loss = weight of edge ij

Minimise loss while meeting the demand.

Variables: P_{ij} = power sent from station i to city j.

$P_{ij} \geq 0$

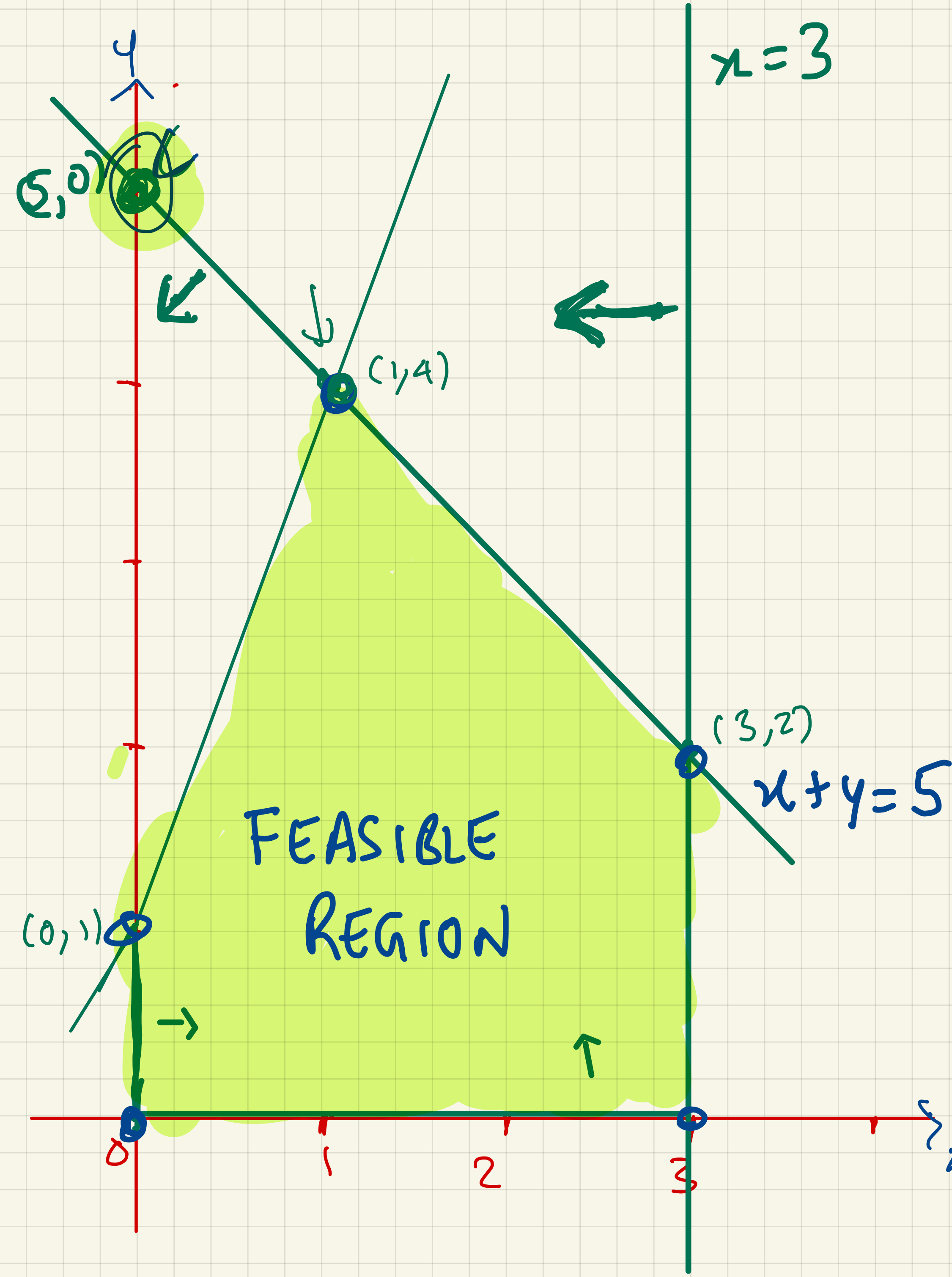
Constraints:

City 1 Demand 40 : $P_{11} + P_{21} \geq 40$

City 2 " 60 : $P_{12} + P_{22} \geq 60$

City " 80 : $P_{13} + P_{23} \geq 80$

Minimize : $4P_{11} + 5P_{12} + P_{13} + 3P_{21} + 2P_{22} + 7P_{23}$



Max $\quad x + 2y$

subject to

A $\quad x \leq 3$

B $\quad (x + y \leq 5)$

C $\quad (y - 3x \leq 1)$

D $\quad x, y \geq 0$

E

TERMINOLOGY:

FEASIBLE: A point $x \in \mathbb{R}^n$ is "feasible" for a linear program, if it satisfies all the constraints

OPTIMAL: A point x is optimal if it minimises/maximizes the objective value

TERMINOLOGY

Polytope = Feasible region of a linear program.

Vertex/Corner = A point x in the feasible region that lies at intersection of "n" hyperplanes (a.k.a. n faces) specified by constraints.

Example: 1) In 2-dimensions, a vertex is intersection of 2 lines

2) In 3-dimensions, a vertex is intersection of 3 faces.

FEASIBLE REGION: Set of points satisfying all constraints.

FACT 1: Feasible region of a linear program is always CONVEX

FACT 2: $\forall$ linear program, there is a optimal solution, which is a "vertex"/corner

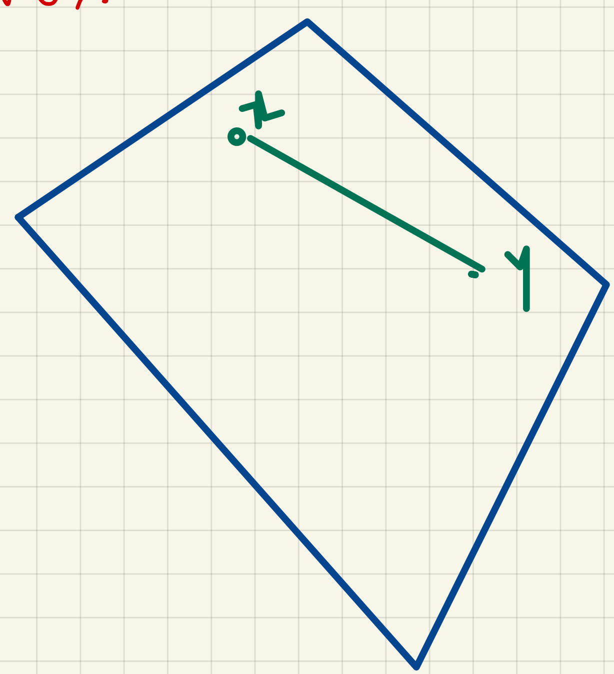
CONVEX SET

A set of points $S \subseteq \mathbb{R}^d$ is convex

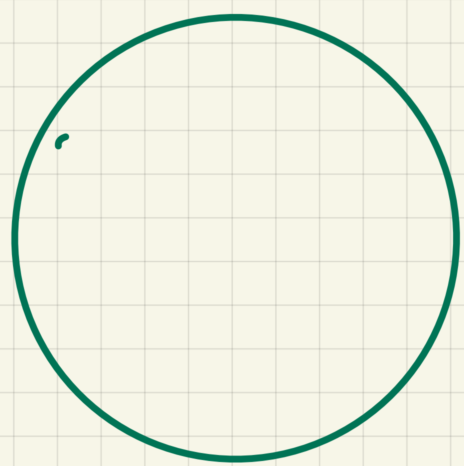
if $\forall\, x, y \in S$

$\implies$ line segment joining $x, y \in S$

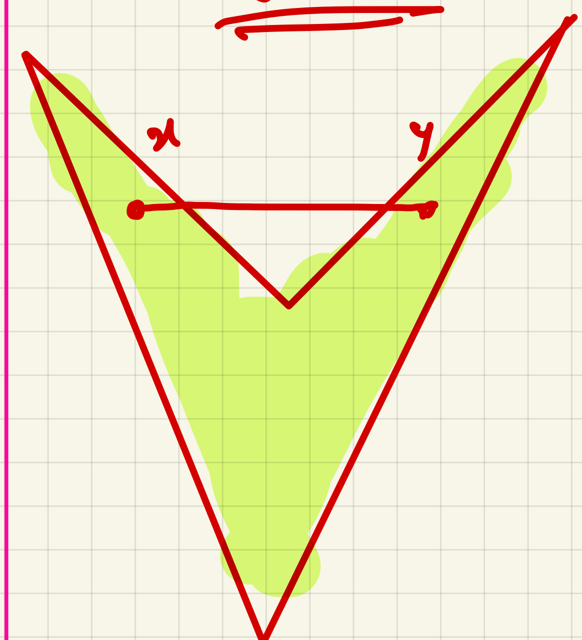
CONVEX

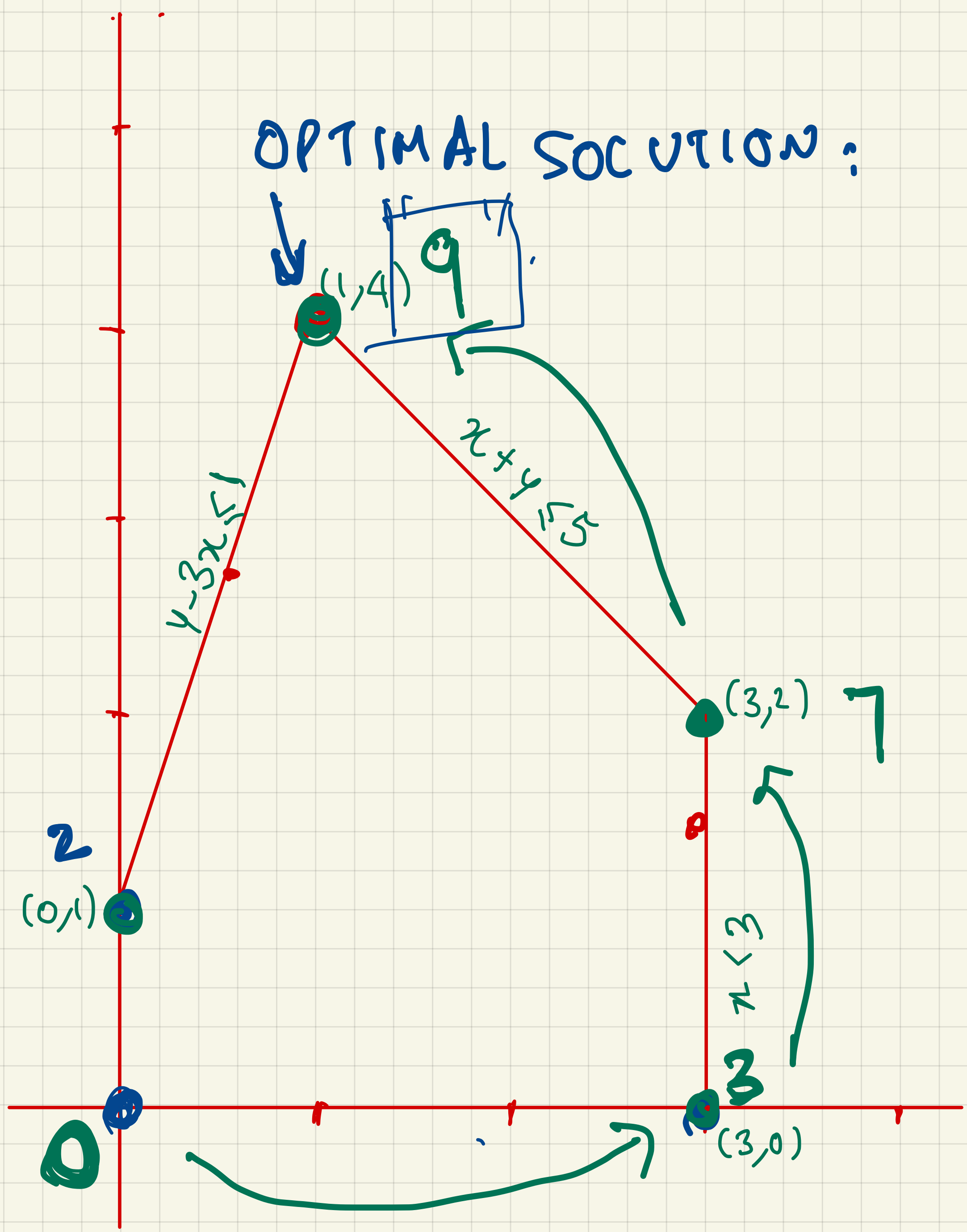


CONVEX



NOT CONVEX





Max $x+2y$

subj. to. $x\leq 3$

A → $(x+y\leq 5)$

B → $(-3x+y\leq 1)$

$x\geq 0$

$y\geq 0$

SIMPLEX ALG:

* Start at some vertex

Ex: $(0,0)$

* Keep moving to neighboring vertex to increase objective

$(0,0)\rightarrow(3,0)\longrightarrow(3,2)\rightarrow(1,4)$

REMARK: 1) Simplex can take exponential time in general, but is very efficient and widely used in practice

2) Linear programs can be solved in polynomial time!

by using a) "Ellipsoid Algorithm"

b) "Interior point methods"

Linear Program

n variables

Variables: $\underbrace{x_1, \dots\ x_n}$ $\in \mathbb{R}^n$

INPUT: $\{a_{ij}, b_j\}$

Constraints:

m constraints

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1 \\ a_{21}x_1 + a_{22}x_2 \qquad + a_{2n}x_n \leq b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 \ \dots \ + a_{mn}x_n \leq b_m \end{cases}$$

Maximise $c_1x_1 + c_2x_2 + \dots c_nx_n$

LISTING ALL "VERTICES" OF A FEASIBLE REGION OF LP

Given an LP $\left\{ \sum a_{ij} x_j \leq b_j : j = 1 \ldots m \right\}$

For each subset of n constraints:

→ Solve for point of intersection x^*

(solving linear system / Gaussian elimination)

→ If x^* is feasible (satisfies all remaining constraint)

then x^* is a vertex.

THEREFORE

$\left\{ \begin{array}{l} \text{\# of vertices of} \\ \text{an LP with} \\ n \text{ variables \& } m \text{ constraints} \end{array} \right\}$ can be as large as $\binom{m}{n} \approx$ exponential in n.

SIMPLEX ALG:

* Start at vertex v
* Find a neighboring vertex of higher objective value and move there, REPEAT.

Number of neighboring vertices

Example: Suppose an LP has 6 constraints with $\{A, B, C, D, E, F\}$ and 3 variables

Consider a vertex := intersection of A, B, C

its neighbors are intersections of

$$\left\{\begin{array}{lll} A, B, D & A, D, C & D, A, C \\ A, B, E & A, E, C & E, A, C \\ A, B, F & A, F, C & F, A, C \end{array}\right\}$$

i.e. Remove one constraint from $\{A, B, C\}$ add one from $\{D, E, F\}$.

In general,

for a vertex v of an LP with n variables and m constraints

$$\#\text{neighbors} \leq \underbrace{(m-n)}_{\text{\# of choices of constraints to add}} \cdot \underbrace{n}_{\text{\# of choices of constraints to remove}}$$

Linear Program

n variables

Variables: $\underbrace{x_1, \ldots \; x_n}_{} \in \mathbb{R}^n$

Constraints:

INPUT: $\{a_{ij}, b_j\}$

m constraints

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\ a_{21}x_1 + a_{22}x_2 \quad\quad + a_{2n}x_n \leq b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 \;\cdots\; + a_{mn}x_n \leq b_m \end{cases}$$

Maximise $c_1x_1 + c_2x_2 + \cdots \; c_nx_n$

CHANGING FORMS OF LP

$\leq$ Constraints to $\geq$ Constraints

$$\sum a_i x_i \geq b_i \iff -\sum_i a_i x_i \leq -b_i$$

$=$ constraints to $\leq$ Constraints

$$\sum a_i x_i = b_i \iff \left\{ \begin{array}{l} \sum a_i x_i \leq b_i \\ -\left(\sum a_i x_i\right) \leq -b_i \end{array} \right\}$$

Maximization to Minimization

$$\text{Max} \sum a_i x_i \iff \text{Min} \; -\sum a_j x_j$$