LECTURE 15

Plan: 1) LINEAR PROGRAMMING

1) DUALITY

2) ZERO SUM GAMES

RECAP:

$$\begin{aligned} \text{Maximize} \quad & 5x + 4y \\ \text{Subject to} \quad & 2x + 3y \leq 10 \\ & x + 2y \leq 20 \\ & x \leq 7 \\ & x, y \geq 0 \end{aligned}$$

Feasible point:

Feasible region:

Optimum :

Edge Cases

$$\boxed{\begin{aligned} \text{Maximise} \quad & 2x + 3y \\ & x + y \leq 10 \\ & x \geq 6 \\ & y \geq 6 \end{aligned}}$$

$$\boxed{\begin{aligned} \text{Maximise} \quad & 2x + 3y \\ & x \leq 10 \end{aligned}}$$

WRITING LINEAR PROGRAMS WITH MATRICES

Variables: $x_1, \dots x_n \in \mathbb{R}^n$

Constraints:

$$\underbrace{\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 \quad + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 \dots + a_{mn}x_n \end{cases}}_{A \cdot x} \begin{matrix} \leq b_1 \\ \leq b_2 \\ \vdots \\ \leq b_m \end{matrix} \quad \leq b$$

Maximise $c_1x_1 + c_2x_2 + \dots c_nx_n$

$$= c^T x$$

LP DUALITY

$$\begin{aligned} 2x_1 + x_2 &\leq 100 \\ x_1 \qquad &\leq 30 \\ x_2 &\leq 60. \end{aligned}$$

$$\boxed{\begin{aligned} x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}}$$

$$5x_1 + 4x_2 \leq 5 \cdot 30 + 4 \cdot 60 = 390$$

Maximize $5x_1 + 4x_2$

OPT ≤ 390.

UPPER BOUNDS

$$\begin{aligned}
(2x_1 + x_2 &\leq 100)\cdot 3 \\
(x_1 &\leq 30)\cdot 0 + \\
(x_2 &\leq 60)\cdot 1
\end{aligned}$$

$x_1 \geq 0$

$x_2 \geq 0$

$$6x_1 + 4x_2 \leq 3 \cdot 100 + 60 = 360$$

$$\begin{matrix} \vee\!| & & \vee\!| \end{matrix}$$

Maximize $5x_1 + 4x_2$ $\qquad$ OPT ≤ 360

$$\begin{aligned}
(2x_1 + x_2 &\leq 100)\cdot 5/2 \\
(x_1 &\leq 30)\cdot 0 \\
(x_2 &\leq 60)\cdot 3/2
\end{aligned}$$

$x_1 \geq 0$

$x_2 \geq 0$

$$5x_1 + 4x_2 \leq (100)\cdot 5/2 + 60 \cdot (3/2)$$

$$= 340$$

Maximize $5x_1 + 4x_2$

$\boxed{\text{OPT} \leq 340}$

LP (PRIMAL)

$$\left(2x_1 + x_2 \leq 100\right) \cdot y_1$$

$$\left(x_1 \leq 30\right) \cdot y_2$$

$$\left(x_2 \leq 60\right) \cdot y_3$$

$x_1 \geq 0$

$x_2 \geq 0$

$$(2y_1 + y_2) \cdot x_1 + (y_1 + y_3) \cdot x_2 \leq 100y_1 + 30y_2 + 60y_3$$

$$\text{Maximize} \quad \underset{\geq}{5}x_1 + \underset{\geq}{4}x_2$$

LP^* (DUAL)

$$\text{Minimise} \quad 100y_1 + 30y_2 + 60y_3$$

$y_1, y_2, y_3 \geq 0$

$$2y_1 + 1 \cdot y_2 + 0 \cdot y_3 \geq 5$$

$$1 \cdot y_1 + 0 \cdot y_2 + 1 \cdot y_3 \geq 4$$

$$5x_1 + 4x_2 \leq (2y_1 + y_2)x_1 + (y_1 + y_3) \cdot x_2$$

$$\leq 100y_1 + 30y_2 + 60y_3$$

PRIMAL LP (Maximization)

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \cdot \begin{bmatrix} \\ x \\ \\ \end{bmatrix} \leq \begin{bmatrix} \\ b \\ \\ \end{bmatrix}$$

$$\text{Max} \begin{bmatrix} & c^T & \end{bmatrix} \begin{bmatrix} \\ x \\ \\ \end{bmatrix}$$

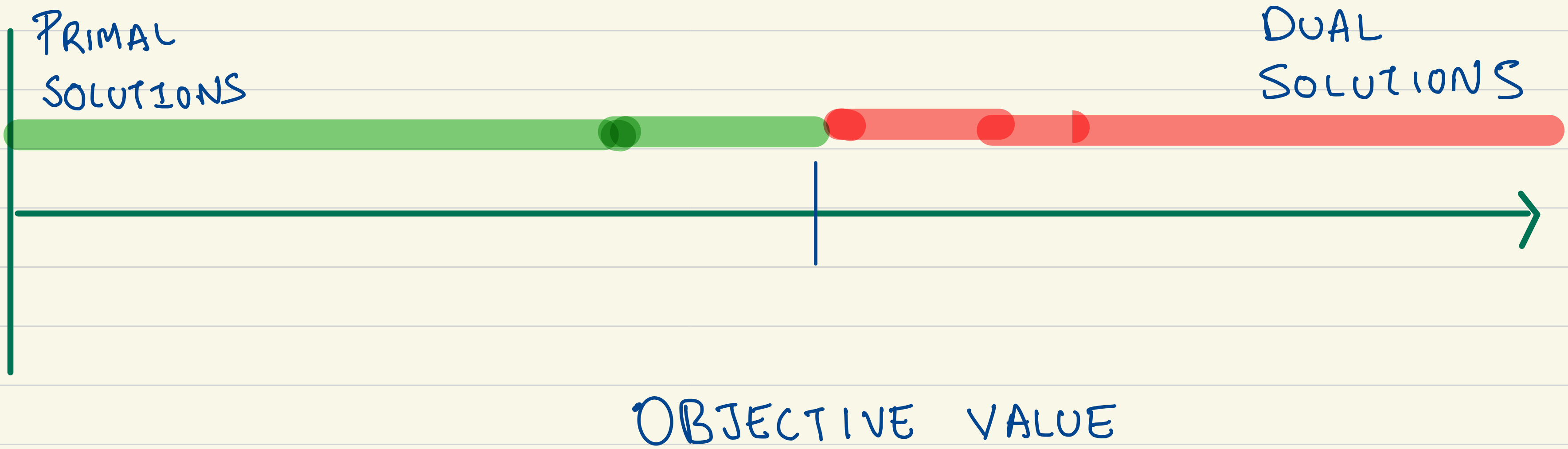
$$\begin{bmatrix} \\ x \\ \\ \end{bmatrix} \geq 0$$

DUAL LP (Minimization)

$$\begin{bmatrix} & & \\ & A^T & \\ & & \end{bmatrix} \cdot \begin{bmatrix} \\ y \\ \\ \end{bmatrix} \geq \begin{bmatrix} \\ c \\ \\ \end{bmatrix}$$

$$\text{Minimise} \begin{bmatrix} & b^T & \end{bmatrix} \begin{bmatrix} \\ y \\ \\ \end{bmatrix}$$

$$\begin{bmatrix} \\ y \\ \\ \end{bmatrix} \geq 0$$



WEAK DUALITY:

$$\left(\text{Any solution to Primal LP}\right) \leq \left(\text{Any solution to Dual LP}\right)$$

STRONG DUALITY:

$$\left(\text{OPTIMAL VALUE OF PRIMAL LP}\right) = \left(\text{OPTIMAL VALUE OF DUAL LP}\right)$$

$$A = \begin{array}{c|ccc} & \text{ROCK} & \text{PAPER} & \text{SCISSOR} \\ \hline \text{ROCK} & 0 & -1 & 1 \\ \text{PAPER} & 1 & 0 & -1 \\ \text{SCISSOR} & -1 & 1 & 0 \end{array}$$

ROW PLAYER

* Picks a row r

COLUMN PLAYER

* Picks a column c

ROW PLAYER WINS A[r,c] $

COLUMN PLAYER LOSES A[r,c] $

VALUE OF GAME = PAYOFF of ROW PLAYER if all play optimally.

GAME A

* ROW PLAYER ANNOUNCES ROW r

* COLUMN PLAYER RESPONDS

	1	2
1	20	-30
2	10	40

GAME B

* COLUMN PLAYER ANNOUNCES COLUMN c

* ROW PLAYER RESPONDS

$$\max_{\text{row } r}\left[\min_{\text{column } c} A[r,c]\right]$$

$$= \max\left[\min_{c} A[1,c],\ \min_{c} A[2,c]\right]$$

$$= \max\left[10,\ -30\right] = 10$$

$$\min_{\text{column } c}\left[\max_{\text{row } r} A[r,c]\right]$$

$$= \min\left[\max_{r} A[r,1],\ \max_{r} A[r,2]\right]$$

$$= \min\left[20,\ 40\right] = 20$$

STRATEGIES:

PURE STRATEGY : Player picks a particular row or column

MIXED STRATEGY : Player picks a probability distribution over rows / columns

Ex: $Pr[row = 1] = 1/2$, $Pr[row = 2] = 1/2$

PAYOFF : Expected payoff of the player

GAME A

* ROW PLAYER ANNOUNCES a mixed strategy

* COLUMN PLAYER RESPONDS with a mixed strategy.

	1	2
1	20	-30
2	10	40

Example:

Row player : $\Pr[\text{row}=1] = 1/4 \quad \Pr[\text{row}=2] = 3/4$

Column player: $\Pr[c=1] = 2/3 \qquad \Pr[c=2] = 1/3$

Expected Payoff

$$= A[1,1]\cdot\left(\tfrac{1}{4}\cdot\tfrac{2}{3}\right) + A[1,2]\cdot\left(\tfrac{1}{4}\cdot\tfrac{1}{3}\right)$$

$$+ A[2,1]\cdot\left(\tfrac{3}{4}\cdot\tfrac{2}{3}\right) + A[2,2]\cdot\left(\tfrac{3}{4}\cdot\tfrac{1}{3}\right)$$

GAME A:

* Row player plays
mixed strategy p

* COLUMN PLAYER responds
mixed strategy q

	1	2
1	20	10
2	10	-30

GAME B

* COLUMN PLAYER plays
mixed strategy q

* ROW PLAYER responds
mixed strategy p

If $p = (p_1, p_2)$ $q = (q_1, q_2)$

EXPECTED PAYOFF $E[p,q] = p_1 \cdot q_1 \cdot A[1,1] + p_1 \cdot q_2 \cdot A[1,2] + p_2 q_1 A[2,1] +$
$p_2 \cdot q_2 \cdot A[2,2]$.

VALUE OF GAME:

A

$$\max_p \left[\min_q E[p,q] \right]$$

(LP)

B

$$\min_p \max_q \left[E[p,q] \right]$$

(LP*)

* LP & LP* are dual of each other.

By strong duality,

$$\mathrm{OPT}(LP) = \mathrm{OPT}(LP^*)$$

$\Rightarrow$ Value of Game A = Value of Game B

$\Rightarrow$ Order of play doesn't matter (with mixed strategies)

WRITING LP for VALUE OF A ZERO-SUM GAME

GAME A:

* Row player plays mixed strategy p
* COLUMN PLAYER responds mixed strategy q

	1	2
1	20	10
2	10	-30

If $p = (p_1, p_2)$ $q = (q_1, q_2)$

EXPECTED PAYOFF $E[p,q] = p_1 \cdot q_1 \cdot A[1,1] + p_1 \cdot q_2 \cdot A[1,2] + p_2 q_1 A[2,1] + p_2 \cdot q_2 \cdot A[2,2]$.

VALUE OF GAME:

$$\max_p \left[\min_q E[p,q] \right]$$

(LP)

FACT:

$$\underset{\substack{\text{ROW MIXED} \\ \text{STRATEGIES} \\ p_1, p_2}}{\text{maximize}} \left[\underset{\substack{\text{COL MIXED} \\ \text{STRATEGIES} \\ q_1, q_2}}{\text{minimize}} \left[20 p_1 q_1 + 10 p_1 q_2 + 10 p_2 q_1 - 30 p_2 q_2 \right] \right]$$

$$= \underset{\substack{\text{ROW MIXED} \\ \text{STRATEGIES} \\ p_1, p_2 \\ p_1 + p_2 = 1}}{\text{maximize}} \left[\underset{\text{COL}}{\text{minimize}} \left[20 p_1 q_1 + 10 p_1 q_2 + 10 p_2 q_1 - 30 p_2 q_2 \right] \right]$$

$$\left\{ \max_{p_1, p_2,\ p_1+p_2=1} \left[\min \left[20p_1 + 10p_2 ,\ 10p_1 - 30p_2 \right] \right] \right\}$$

$$\Updownarrow$$

LP

$$\boxed{\begin{array}{l} \text{Maximije} \quad z \\ z \leq 20p_1 + 10p_2 \\ z \leq 10p_1 - 30p_2 \\ p_1 + p_2 = 1 \end{array}}$$

$$\parallel$$

VALUE OF GAME A

GAME B

$$\min_{\substack{\text{COL MIXED STRATEGIES} \\ q_1, q_2}} \left[\max_{\substack{\text{ROW MIXED STRATEGIES} \\ p_1, p_2}} \left[20 p_1 q_1 + 10 p_1 q_2 + 10 p_2 q_1 - 30 p_2 q_2 \right] \right]$$

$$= \min_{\substack{\text{COL MIXED STRATEGIES} \\ q_1, q_2}} \left[\max_{\substack{\text{ROW PURE STRATEGIES} \\ p_1 = 0 \; p_2 = 1 \\ p_1 = 1 \; p_2 = 0}} \left[20 p_1 q_1 + 10 p_1 q_2 + 10 p_2 q_1 - 30 p_2 q_2 \right] \right]$$

$$= \min_{q_1, q_2, \, q_1 + q_2 = 1} \left[\max \{ 20 q_1 + 10 q_2 , \, 10 q_1 - 30 q_2 \} \right]$$

LP^*

$$\begin{aligned} \text{Min} \quad & z \\ & 20q_1 + 10q_2 \leq z \\ & 10q_1 - 30q_2 \leq z \\ & q_1 + q_2 = 1 \\ & q_1, q_2 \geq 0 \end{aligned}$$