LECTURE 16

1) ZERO-SUM GAMES.

   - Writing LP for optimal strategy

2) Maximum Flow.

WRITING LP for VALUE OF A ZERO-SUM GAME

GAME A:

* Row player plays mixed strategy p
* COLUMN PLAYER responds mixed strategy q

	1	2
1	20	10
2	10	-30

If $p = (p_1, p_2)$ $q = (q_1, q_2)$

EXPECTED PAYOFF $E[p,q] = p_1 \cdot q_1 \cdot A[1,1] + p_1 \cdot q_2 \cdot A[1,2] + p_2 q_1 A[2,1] + p_2 \cdot q_2 \cdot A[2,2]$.

VALUE OF GAME:

$$\max_p \left[\min_q E[p,q] \right]$$

(LP)

FACT:

$$\underset{\substack{\text{(ROW, MIXED} \\ \text{STRATEGIES)} \\ p_1, p_2}}{\text{maximize}} \left[\underset{\substack{\text{(COL MIXED} \\ \text{STRATEGIES)} \\ q_1, q_2}}{\text{minimize}} \left[\begin{array}{l} 20 p_1 q_1 + 10 p_1 q_2 \\ +10 p_2 q_1 - 30 p_2 q_2 \end{array} \right] \right]$$

$$= \underset{\substack{\text{(ROW, MIXED} \\ \text{STRATEGIES)} \\ p_1, p_2 \\ p_1 + p_2 = 1}}{\text{maximize}} \left[\underset{\text{(COL }}{\text{minimize}} \quad \left[\begin{array}{l} 20 p_1 q_1 + 10 p_1 q_2 \\ +10 p_2 q_1 - 30 p_2 q_2 \end{array} \right] \right]$$

$$\left\{ \max_{p_1, p_2,\ p_1+p_2=1} \left[\min \left[20p_1 + 10p_2,\ 10p_1 - 30p_2 \right] \right] \right\}$$

$$\Updownarrow$$

LP

$$\boxed{\begin{aligned} &\text{Maximize } z \\ &z \leq 20p_1 + 10p_2 \\ &z \leq 10p_1 - 30p_2 \\ &p_1 + p_2 = 1 \end{aligned}}$$

$$\parallel$$

VALUE OF GAME A

GAME B

$$\min_{\substack{\text{COL MIXED STRATEGIES} \\ q_1, q_2}} \left[\max_{\substack{\text{ROW MIXED STRATEGIES} \\ p_1, p_2}} \left[20p_1q_1 + 10p_1q_2 + 10p_2q_1 - 30p_2q_2 \right] \right]$$

$$= \min_{\substack{\text{COL MIXED STRATEGIES} \\ q_1, q_2}} \left[\max_{\substack{\text{ROW PURE STRATEGIES} \\ p_1=0\ p_2=1 \\ p_1=1\ p_2=0}} \left[20p_1q_1 + 10p_1q_2 + 10p_2q_1 - 30p_2q_2 \right] \right]$$

$$= \min_{q_1, q_2,\, q_1+q_2=1} \left[\max\{20q_1+10q_2,\ 10q_1-30q_2\} \right]$$

LP^*

$$\begin{aligned} \text{Min} \quad & z \\ & 20q_1 + 10q_2 \le z \\ & 10q_1 - 30q_2 \le z \\ & q_1 + q_2 = 1 \\ & q_1, q_2 \ge 0 \end{aligned}$$

MAXIMUM FLOW

Maximum Flow

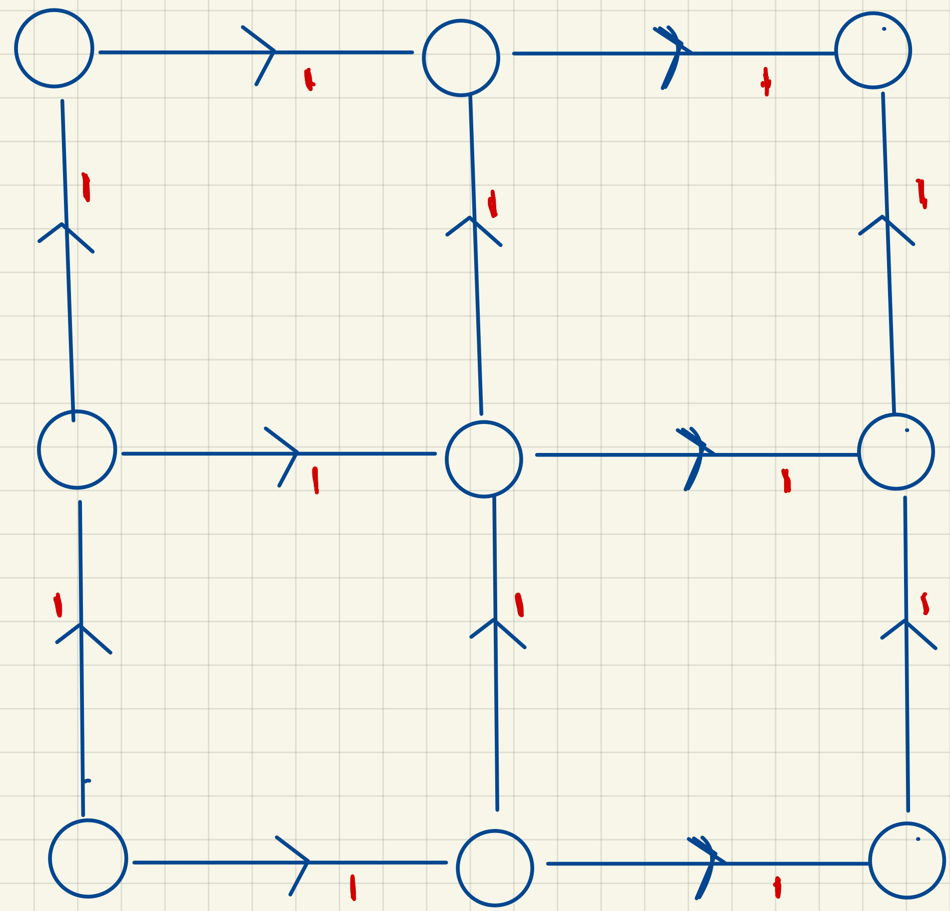
Setup: 1) Directed graph

$G = (V, E)$

2) "Source" s node

3) "Sink" t node

4) Capacities $c_e \in \mathbb{R}^+$ $\mathbb{Z}^+$ for each edge e



Definition (s-t-Flow)

An s-t flow is an assignment $f: E \to \mathbb{R}^+$ such that

1) Capacity constraint: For each edge e,

flow on e $\leqslant$ capacity of e

$$f_e \leqslant c_e$$

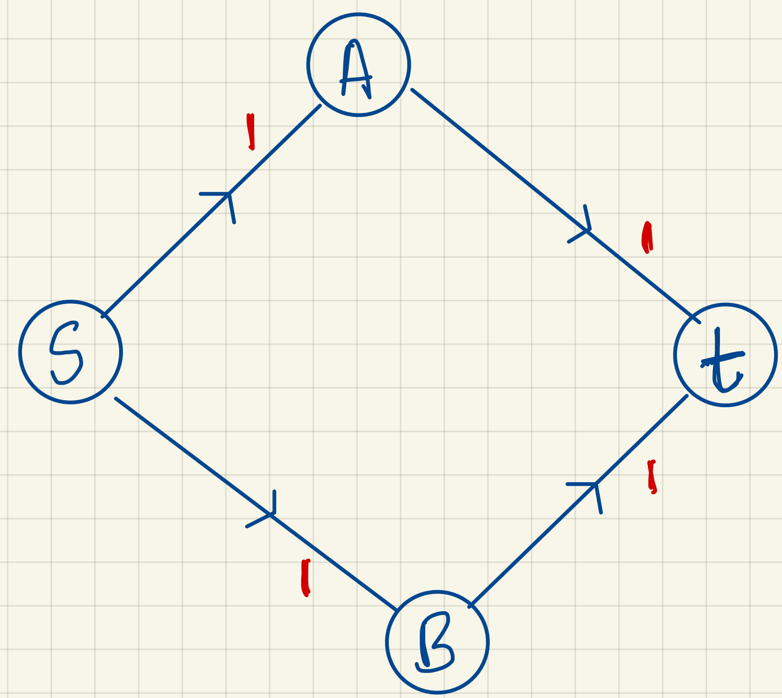
2) Conservation constraint: For each vertex $v \neq s/t$

Flow coming into v = Flow leaving v

$$\sum_{v \to u} f_{v \to u} = \sum_{u \to w} f_{u \to w}$$

Max s-t Flow: maximize flow leaving s $= \sum_{u} f_{s \to u}$

ALGORITHM TO COMPUTE MAX-FLOW (A SKETCH)

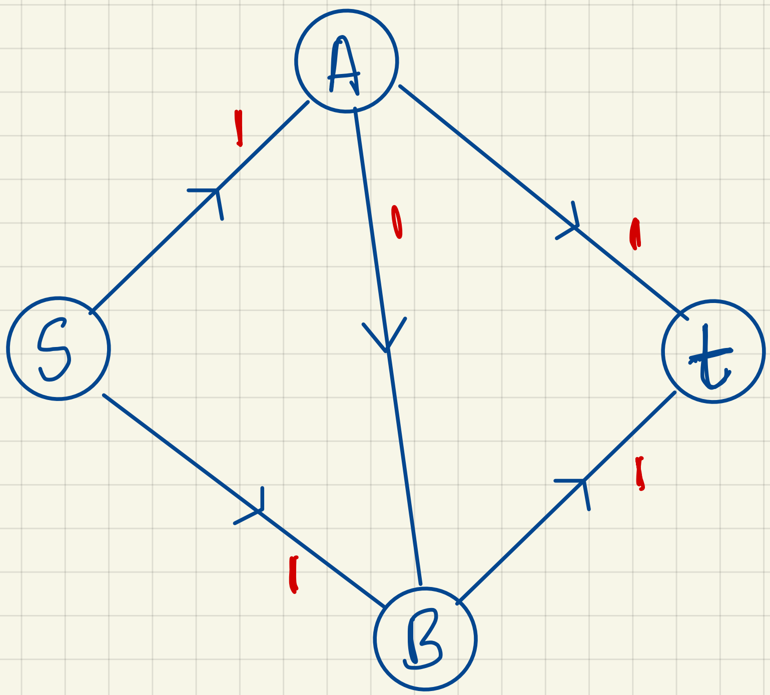


HIGH-LEVEL SKETCH

REPEAT: 1) Find a path P from s to t with left-over capacity to send more flow.

2) Add flow along P.

FAILURE OF ALGO



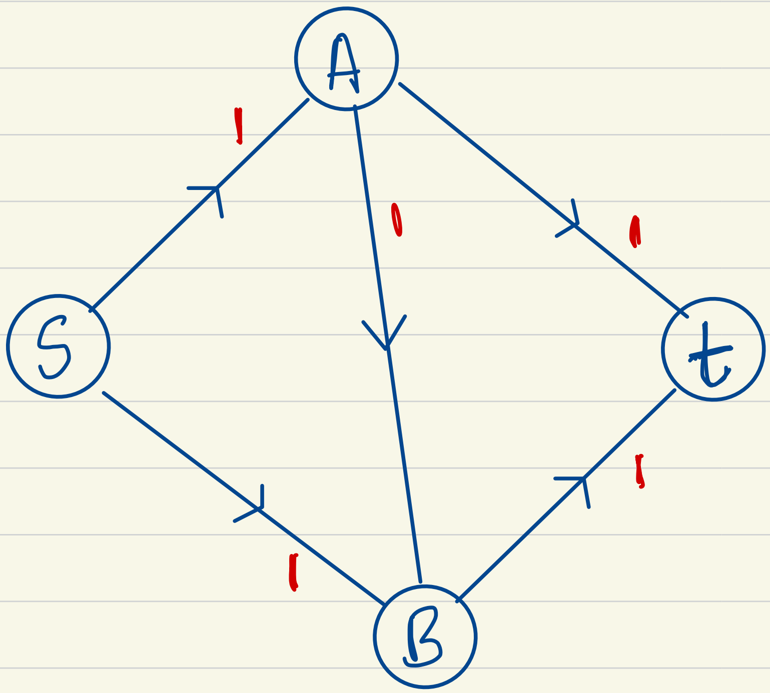
ALGO FOR MAX FLOW

REPEAT:

* FIND A PATH P from s to t with non-zero capacity in RESIDUAL GRAPH

[TERMINATE if NO PATH P exists]

* Add flow along P to the current flow


A
S
t
B
1
1
0
1
1

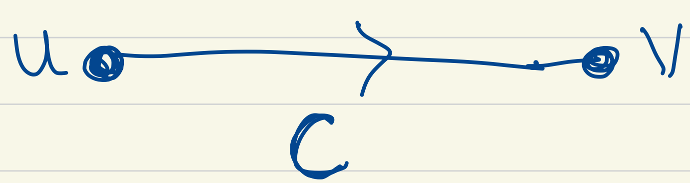
Residual Graph:

Given: * $G = (V, E)$ is a directed graph

* f is some flow on G.

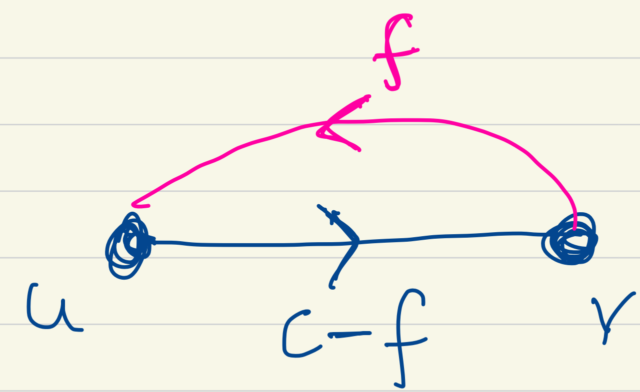
THE RESIDUAL GRAPH G_f on same vertices V and edges

$\forall$ edge u, v with capacity c flow f



in ORIGINAL GRAPH G

$\Longrightarrow$



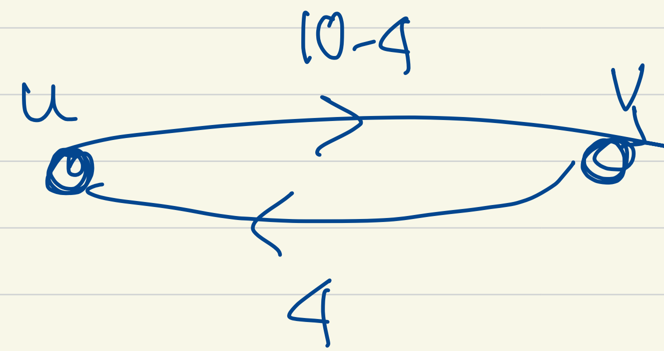
in RESIDUAL GRAPH

EXAMPLE:

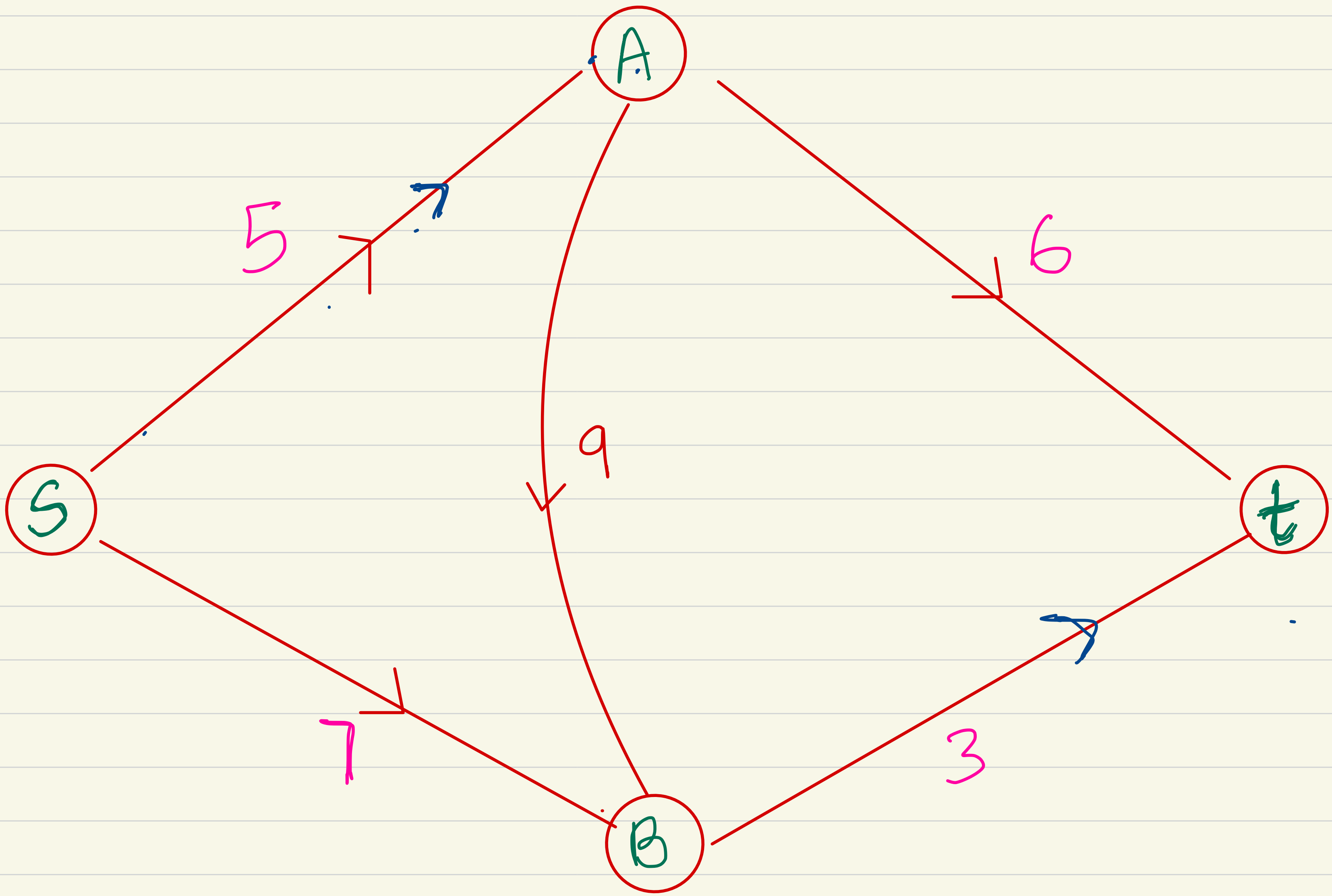
ORIGINAL GRAPH

Capacity 10
Flow 4

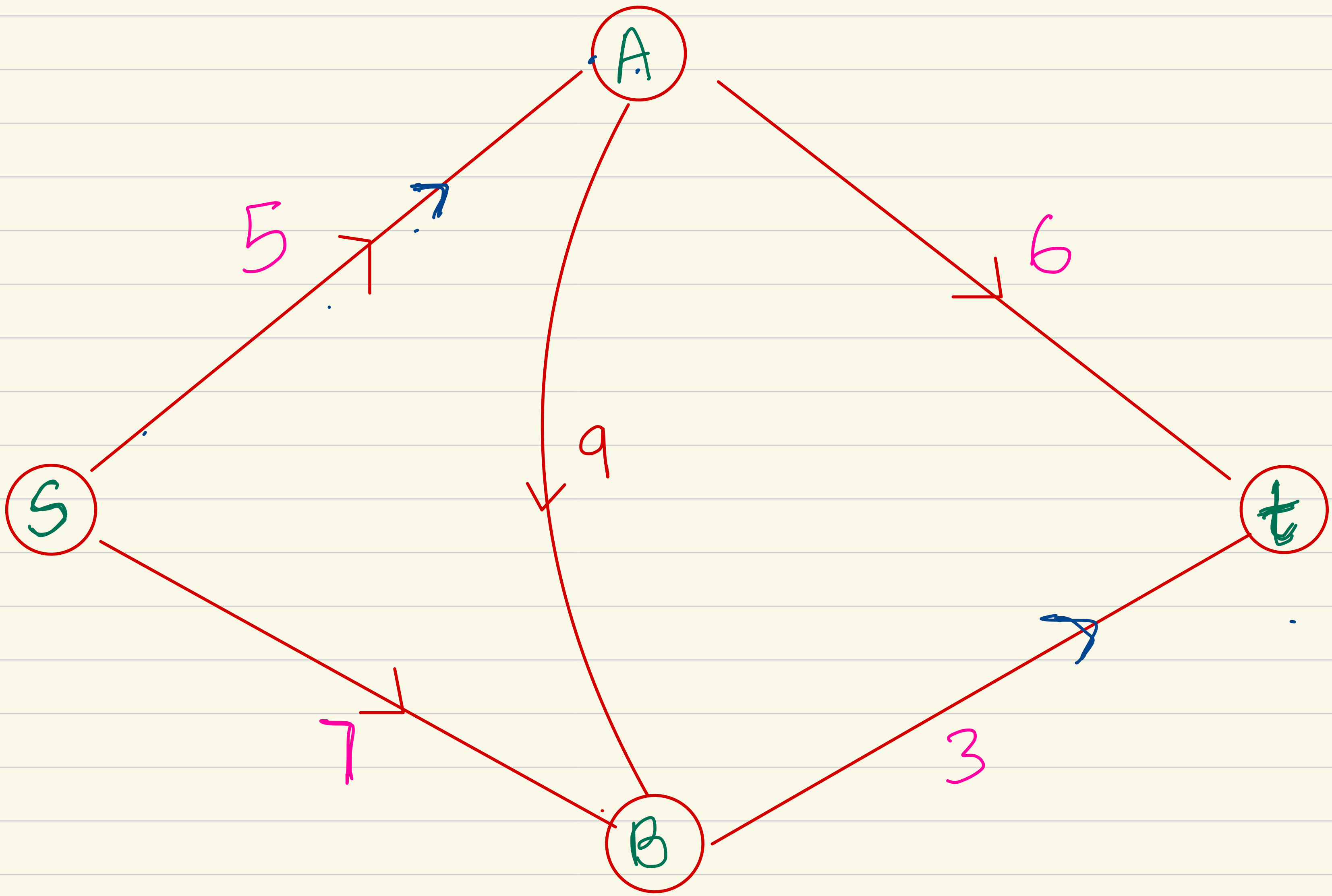
RESIDUAL GRAPH



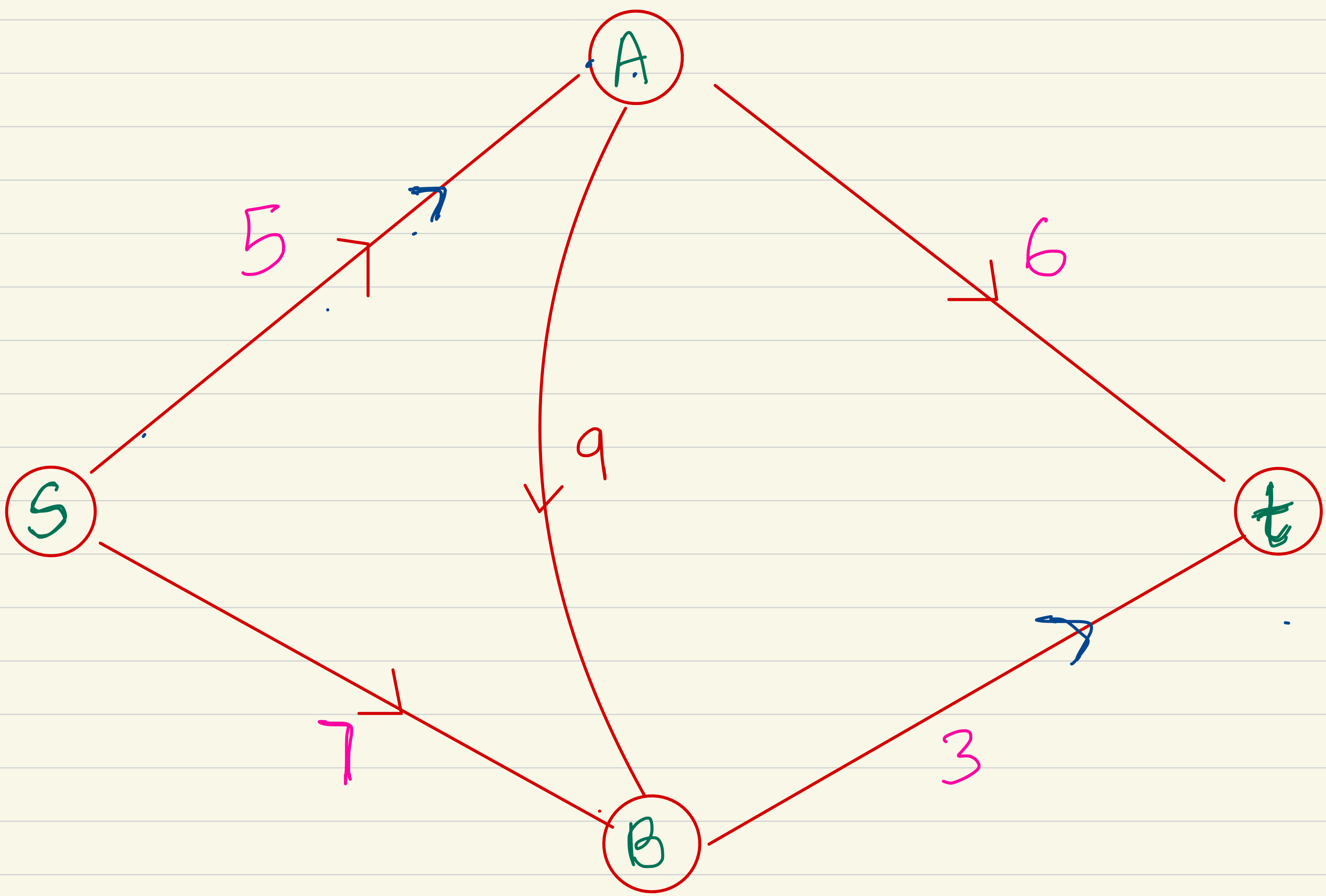
MAX FLOW EXECUTION



MAX FLOW EXECUTION



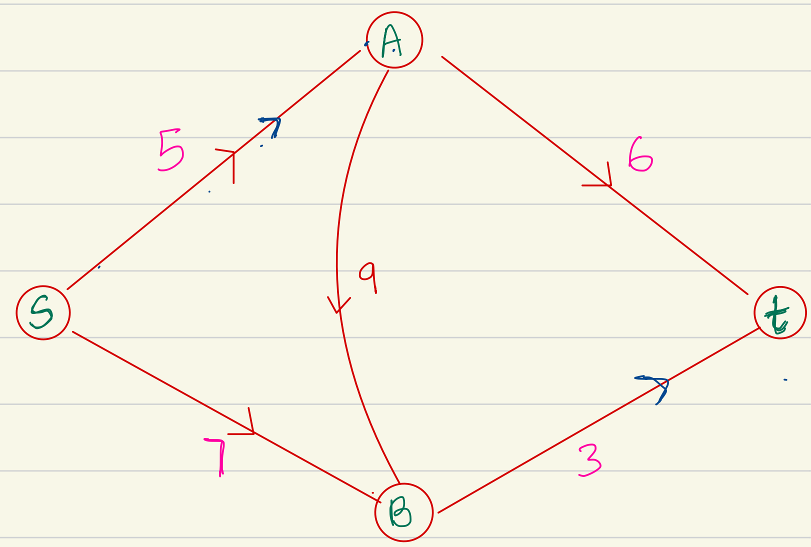
CAN THE FLOW BE HIGHER THAN 8 ?¿



s-t Cuts

DEFINITION (s-t CUT): A partition $V = L \cup R$ so that $s \in L$, $t \in R$

Example:



$L = \{s\}$ $R = \{A, B, t\}$

$L = \{s, A\}$ $R = \{B, t\}$

DEFINITION (CAPACITY OF AN s-t CUT)

capacity (L, R) = Total capacity of edges from L to R

$$= \sum_{\substack{u \to v \\ u \in L \ v \in R}} c_{u \to v}$$

<u>CLAIM:</u> For every s-t cut L,R and for every s-t flow f

$$\text{Size}(f) \leq \text{CAPACITY}(L, R)$$

$$\Rightarrow \quad \text{Maximum Flow} \leq \text{Minimum Cut}$$

THEOREM: IN ANY GRAPH G,

$$\text{MAXIMUM } s\text{-}t \text{ FLOW} = \text{MINIMUM } s\text{-}t \text{ CUT}$$

PROOF:

Execute algorithm to compute maximum flow

At the end, define

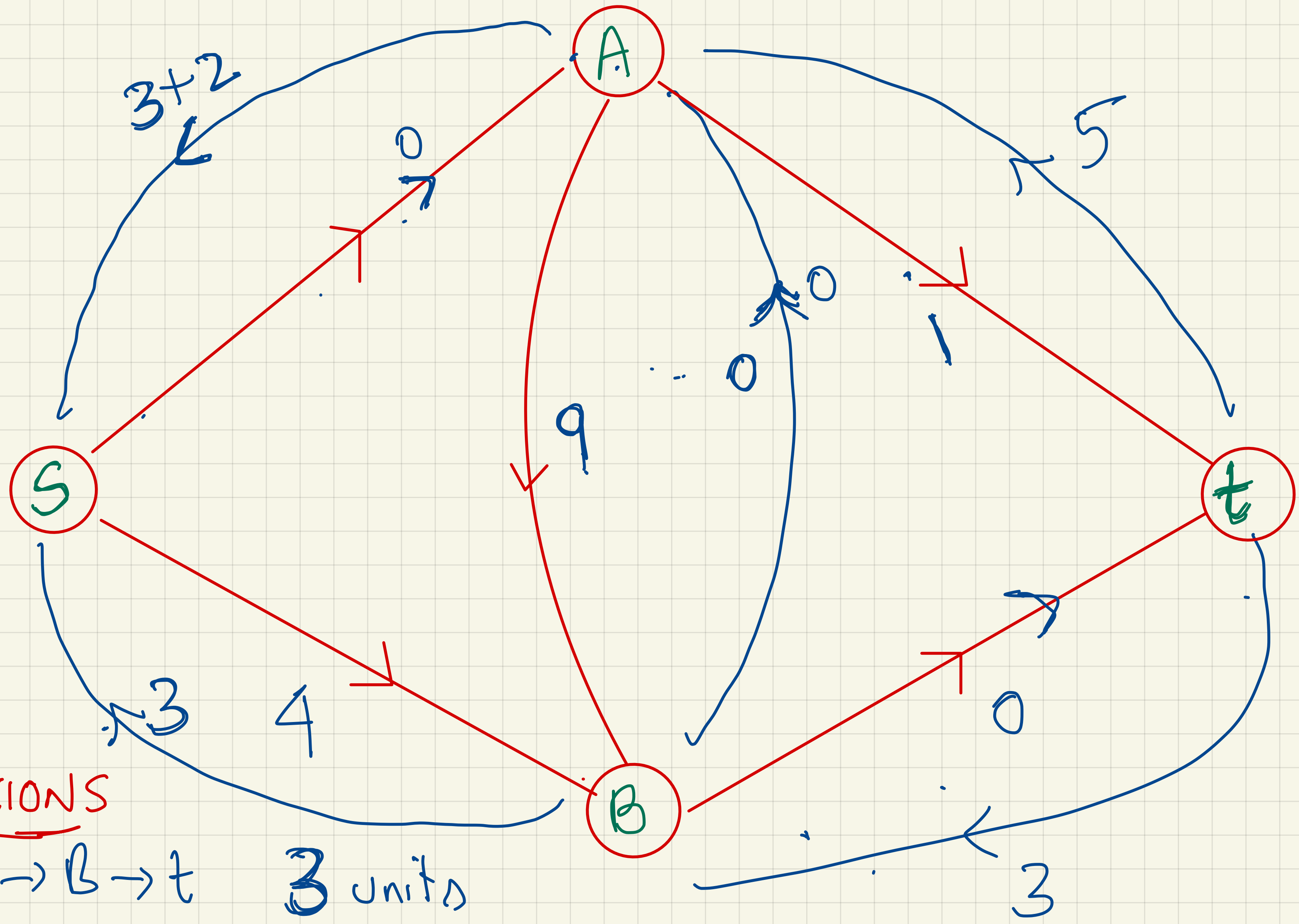
$$L = \{\text{set of vertices reachable from } s \text{ in residual graph } G_f\}$$

$$R = V \setminus L$$

$\nexists$ no residual capacity leaving L

$\Rightarrow$ All edges leaving L are saturated

$\Rightarrow$ Capacity (L, R) = Total flow leaving L / s



ITERATIONS

$S \to A \to B \to t$ 3 units

$S \to A \to t$ 2 units

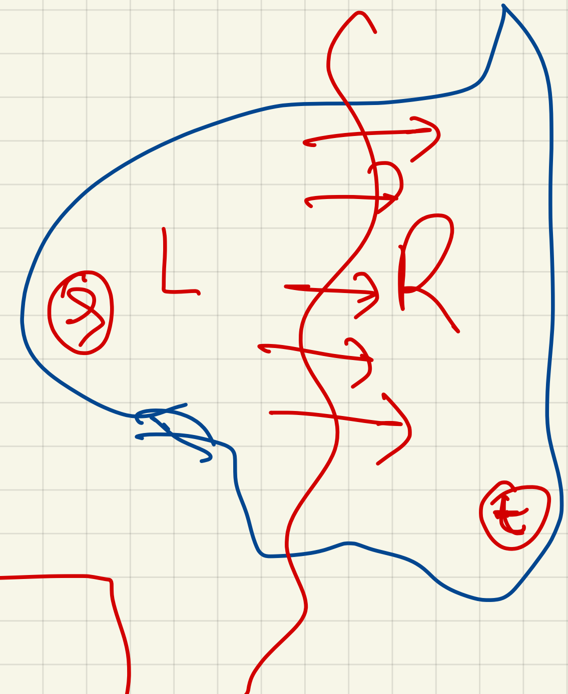
$S \to B \to A \to t$ 3 units

s-t Cut: An s-t Cut is (L, R)

$$L \cup R = V$$

L → left, R → right

$s \in L \quad t \in R$

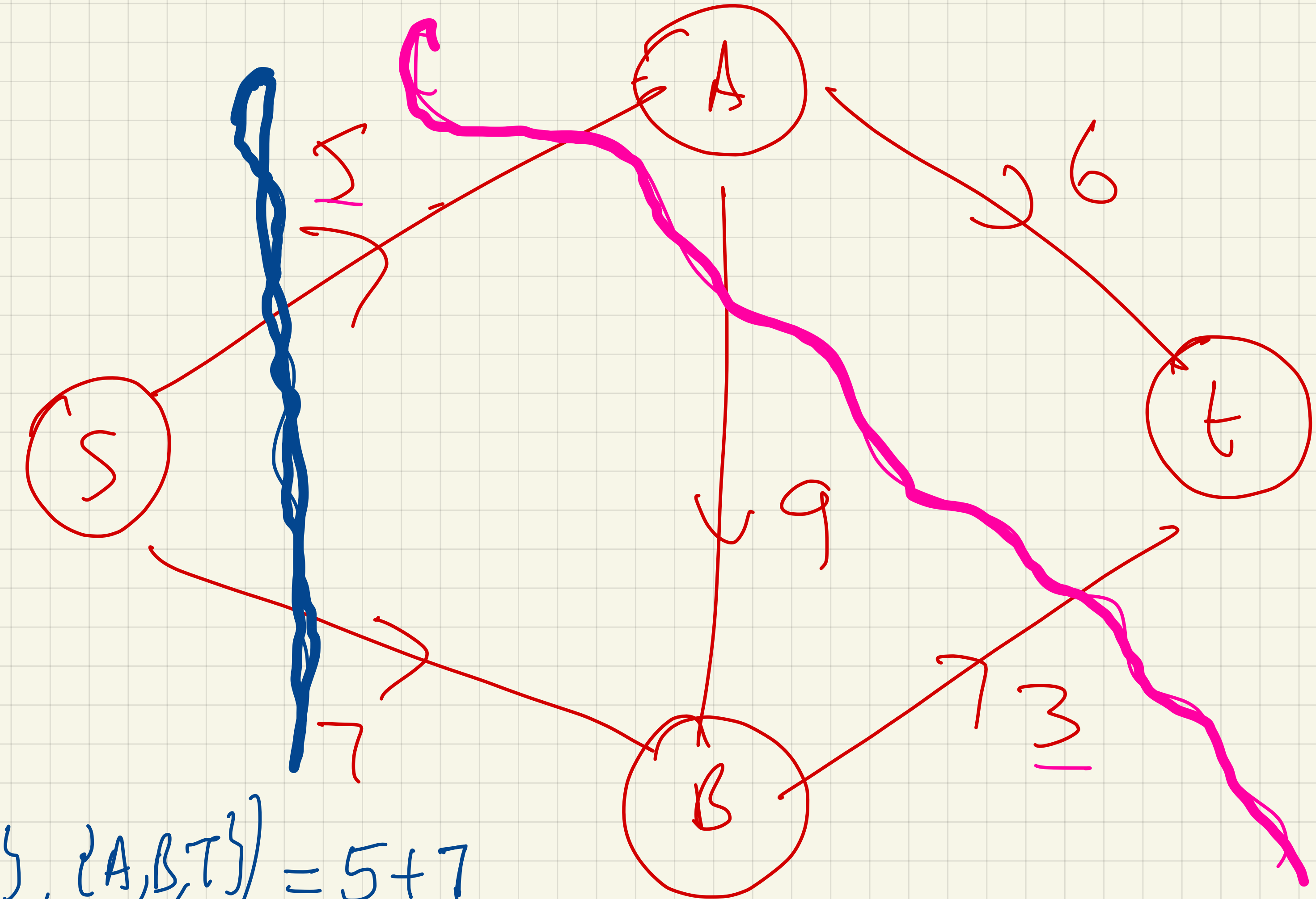


$$\boxed{\text{capacity}(L,R) = \sum_{\substack{u \to v \\ u \in L,\ v \in R}} c_{u \to v}}$$

FACT: For any flow f, and any cut (L,R)

$$\text{size}(f) \leq \text{capacity}(L,R)$$

CUTS:



EXAMPLES:

$$\text{Capacity}(\{S\}, \{A, B, T\}) = 5 + 7$$
$$= 12$$

$$\text{Capacity}(\{S, B\}, \{A, t\}) = 5 + 3 = 8$$