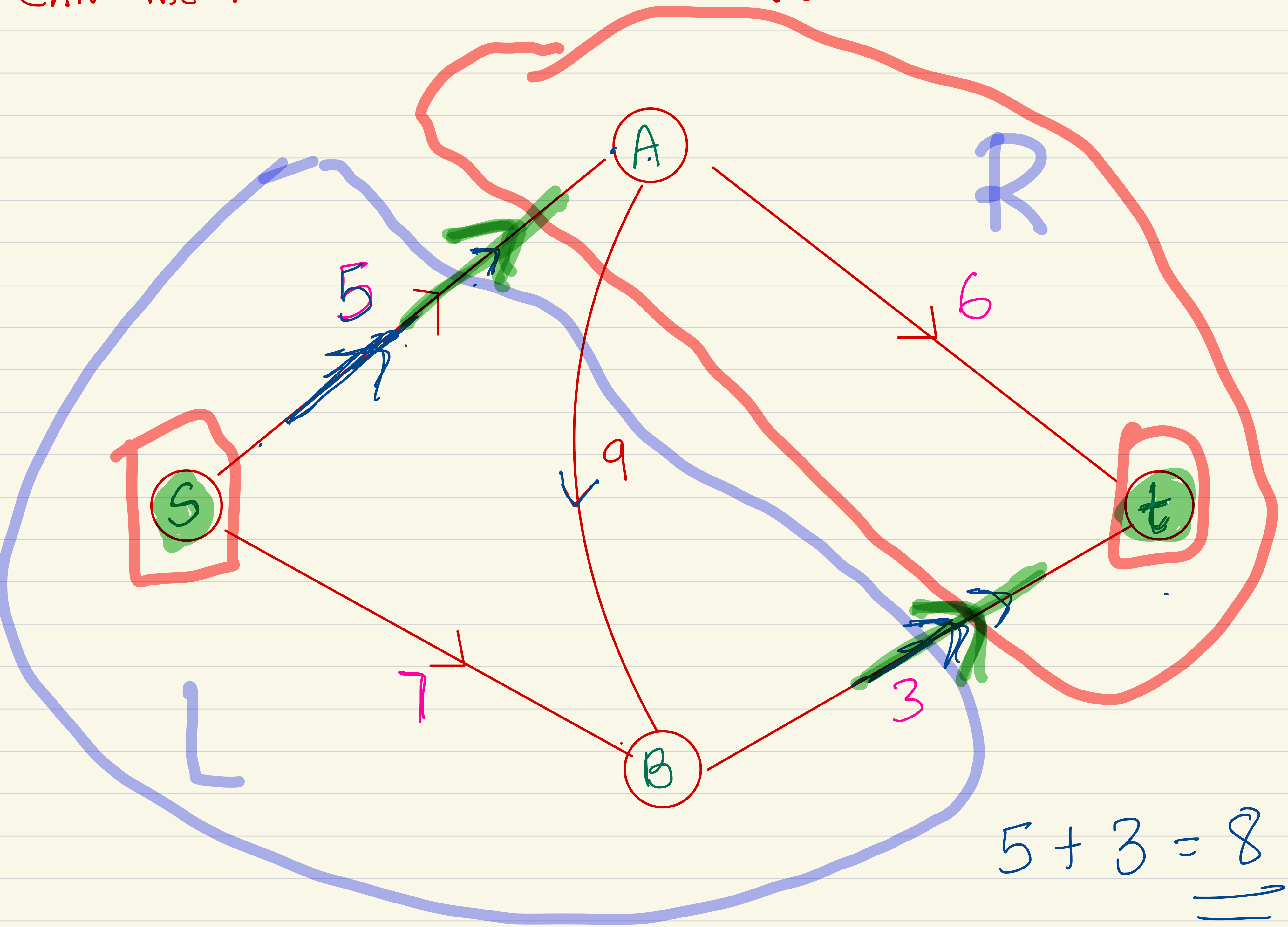
$R = V \setminus L$

LECTURE 17

PLAN:

1) ANALYZE our algo for Max Flow
2) Max-Flow MinCut Theorem
3) Application: Bipartite Matching
4) Solving LPs

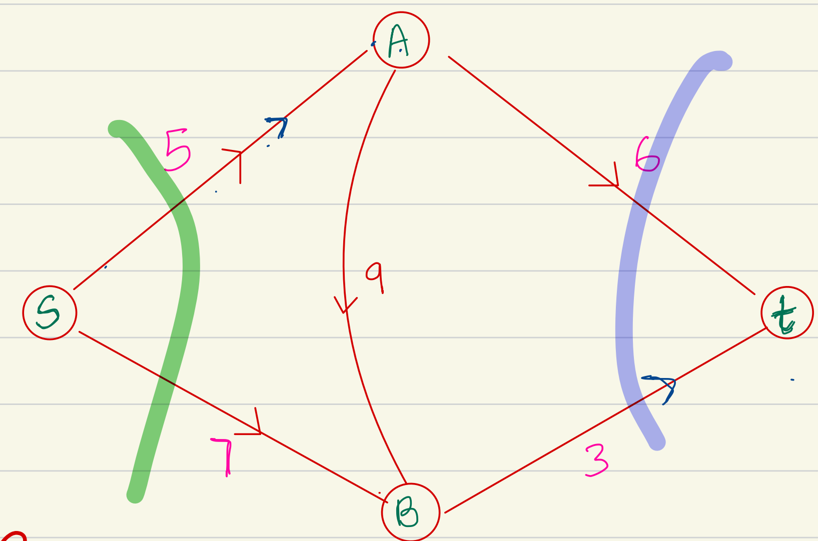
CAN THE FLOW BE HIGHER THAN 8 ??



$5 + 3 = \underline{\underline{8}}$

s-t Cuts

DEFINITION (s-t CUT): A partition $V = L \cup R$ so that $s \in L$ $t \in R$



$L = \{s\}$ $R = \{t, A, B\}$ $5+7=12$

$L = \{s, A, B\}$ $R = \{t\}$ $6+3=9$

$L = \{s, B\}$ $R = \{t, A\}$ $5+3=8$

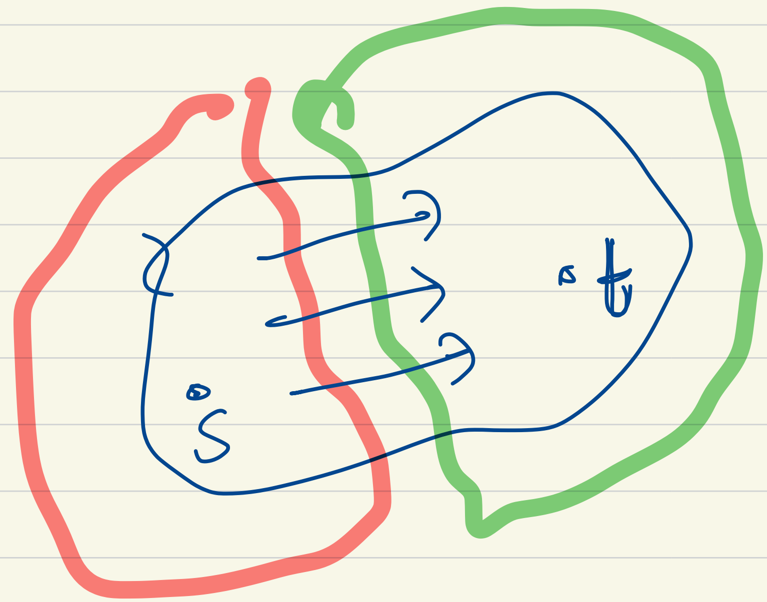
DEFINITION (CAPACITY OF AN s-t CUT)

capacity(L, R) = Total capacity of edges from L to R

$$= \sum_{\substack{u \to v \\ u \in L,\ v \in R}} c_{u \to v}$$

$\boxed{\text{CLAIM 1}}$ In any graph

EVERY s-t FLOW $\leq$ CAPACITY OF EVERY s-t CUT



Proof:

THEOREM: IN ANY GRAPH G,

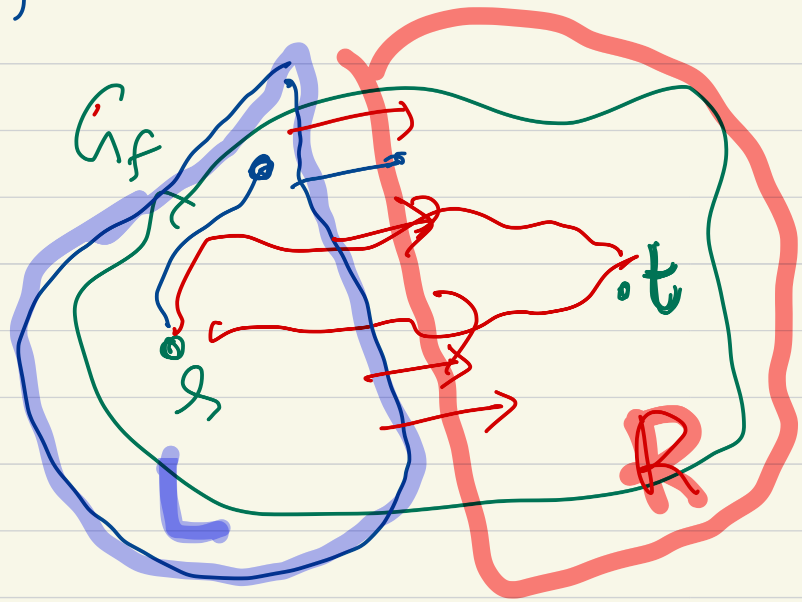
MAXIMUM s-t FLOW $=$ MINIMUM s-t CUT

PROOF:

1) Execute the algorithm for Max Flow

At termination:

There is NO s-t path in the residual graph G_f



Consider

$$L = \{\text{set of vertices reachable by a path from } s \text{ in } G_f\}$$

$R = V \setminus L$

$\nexists$ no edge from L to R in G_f

residual graph

$\iff$ Every edge from L to R in G is <u>saturated</u>

$\longrightarrow$

capacity $= 3$

flow $= 3$

$\iff$ $\forall$ edge e from L to R

$$f_e = \text{capacity}_e$$

$\Rightarrow$ Total flow from L to R $= \sum_{e: L \text{ to } R}$ capacity$_e$

Conclusions: i) At termination, $\exists$ a cut with value $=$ flow assigned

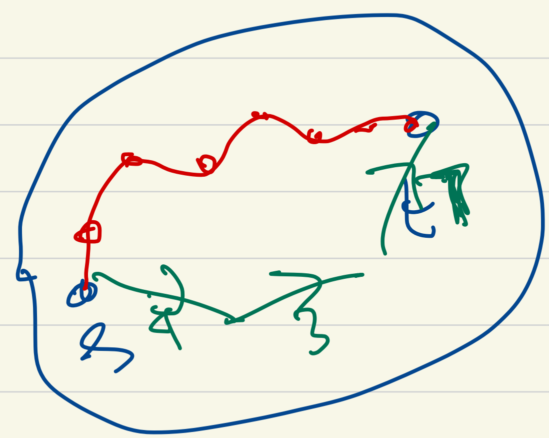
Since All flows $\leq$ Capacity of all cuts

$\Rightarrow$ At termination, current flow $=$ Max flow.

COROLLARY [OF MAX-FLOW ALGORITHM]:

In a network $G = (V, E)$ if all capacities are <u>integers</u> then $\exists$ a maximum flow solution which is also integral.

REMARK: In general, optimal CP solutions need not be integral.

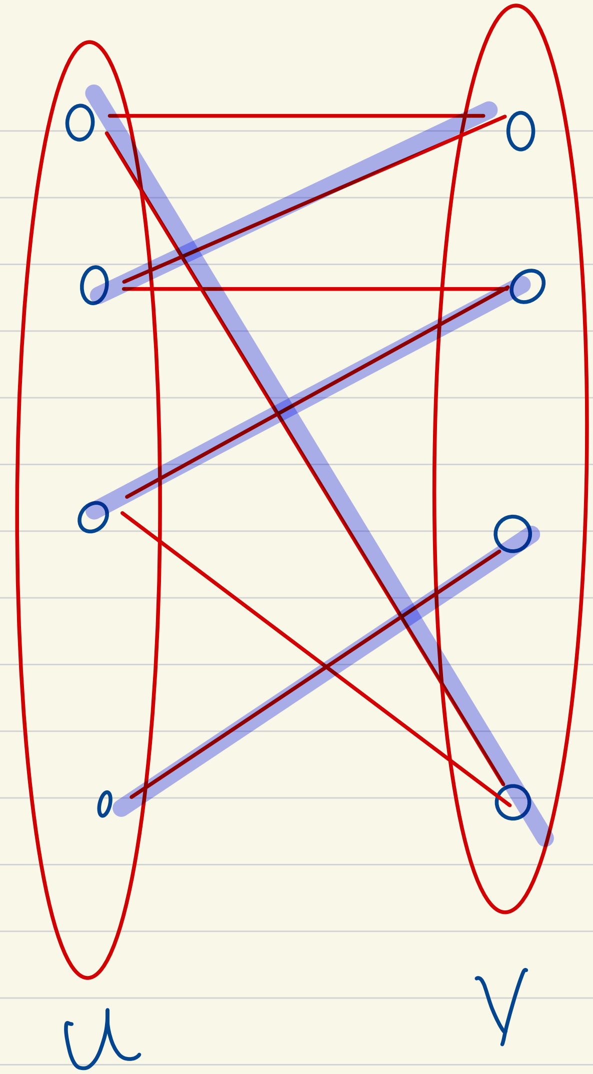


PERFECT MATCHING:

INPUT: Bipartite Graph

$|U| = |V| = n$ $(U \cup V, E)$

GOAL: Find a perfect matching from U to V.



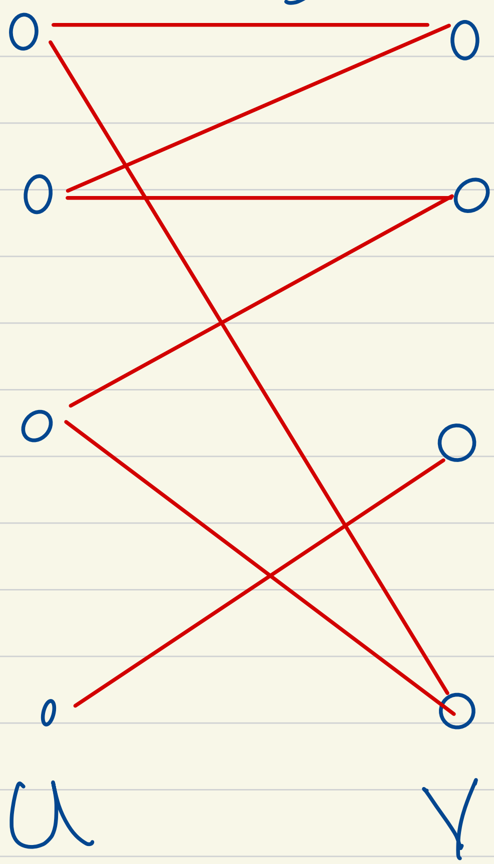
(non-overlapping)

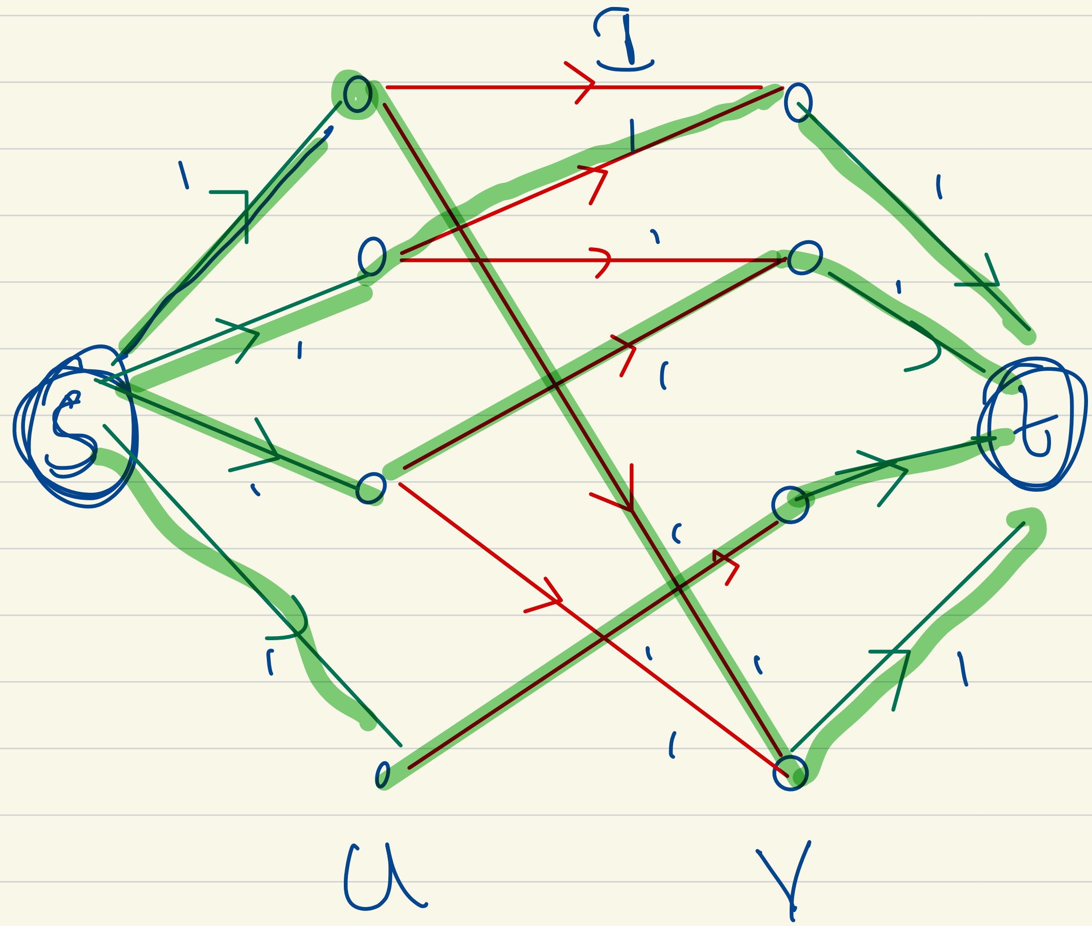
Matching = A set of disjoint pairs

"Perfect" Matching = Every vertex is matched.

PERFECT MATCHING $\longrightarrow$ MAXIMUM FLOW

unweighted





1) Compute Maximum s-t Flow

Maximum Flow Value $= n \ (n=4) \iff \exists$ a perfect matching

ASSIGNMENT PROBLEMS:

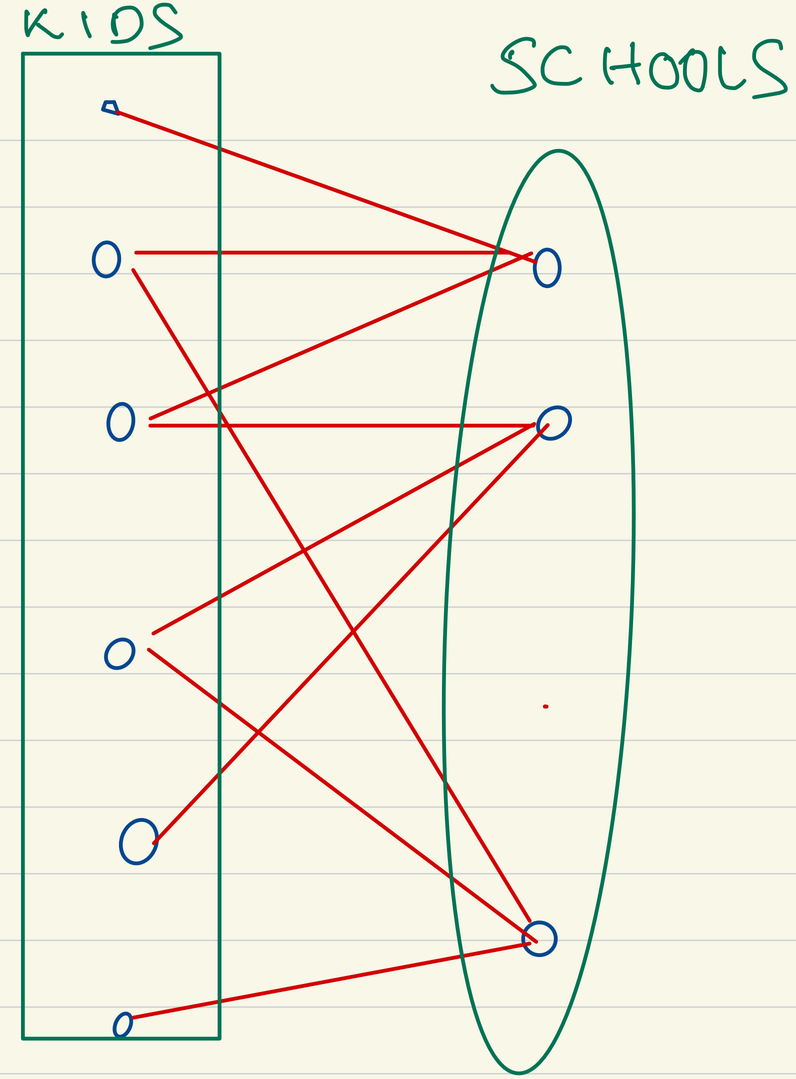
INPUT:

1) n schools with capacity $c_1, \ldots, c_n$

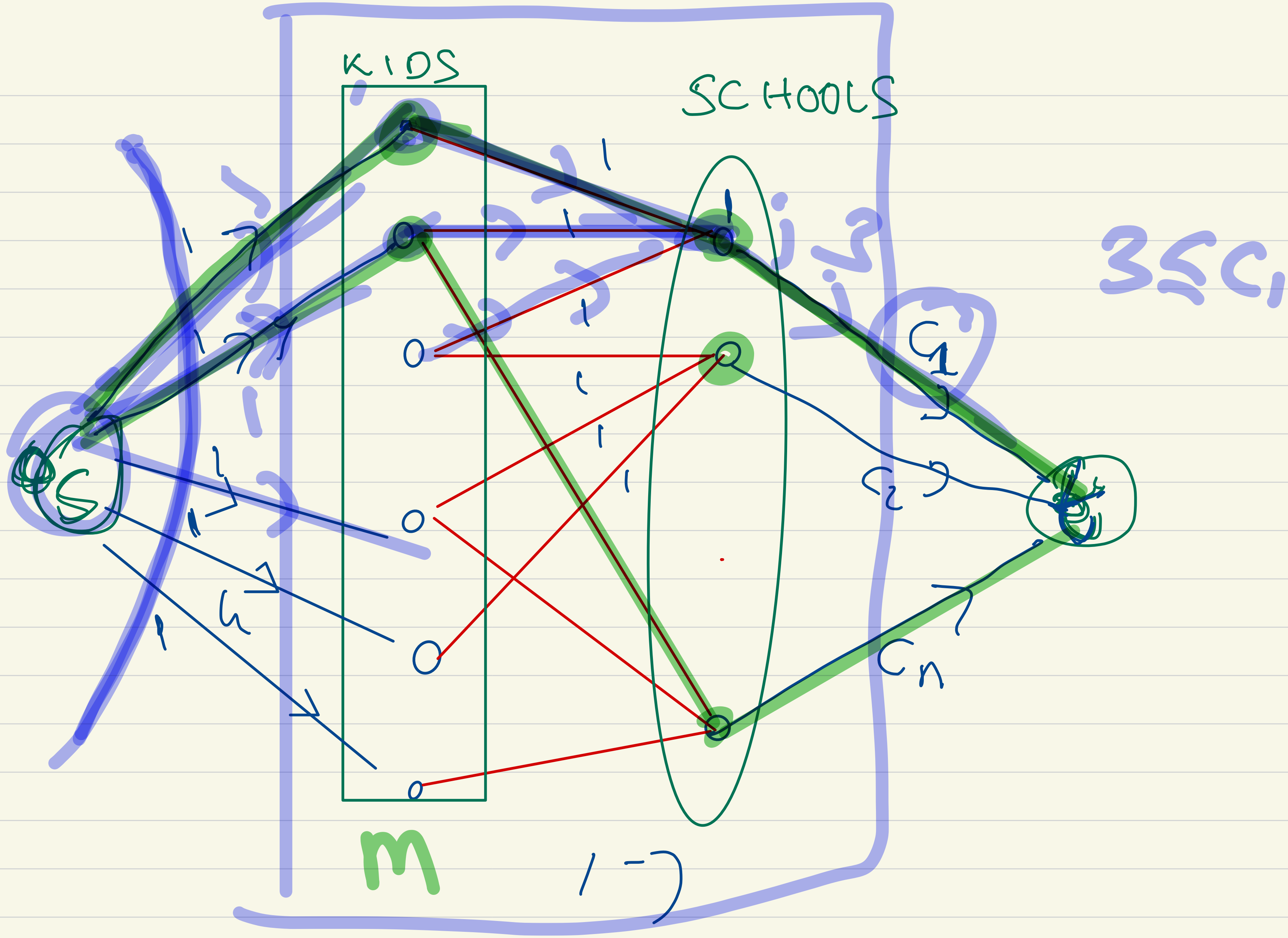
2) m children

Graph $(\mathcal{U} \cup V, E)$ is such that

(i,j) is an edge if child i can go to school j



$(\mathcal{U} \cup V, E)$


KIDS
SCHOOLS
m
S
t
c_1
c_2
c_n
$3 \le c_1$

SOLVING LPs

VIA

"GRADIENT DESCENT"

OPTIMIZATION vs

$$\boxed{\begin{aligned}&\text{Maximise } c^T x\\&\text{Subj to } Ax \leq b\end{aligned}}$$

Given constraints find x that is optimal while satisfying constraints.

FEASIBILITY

$$\boxed{\begin{aligned}&\text{Find } x \text{ satisfying}\\&Px \leq q\end{aligned}}$$

Find x satisfying all constraints.

Thm: An algorithm for FEASIBILITY OF LPs

$\implies$ An algorithm for OPTIMIZATION OF LPs

Proof:

OPT:

$$\boxed{\begin{aligned} &\text{Maximize} && x + 2y \\ &\text{Subj} && 2x + 9y \le 6 \\ & && 3x + 2y \le 7 \\ & && x, y \ge 0 \end{aligned}}$$

FEASIBLE(C)

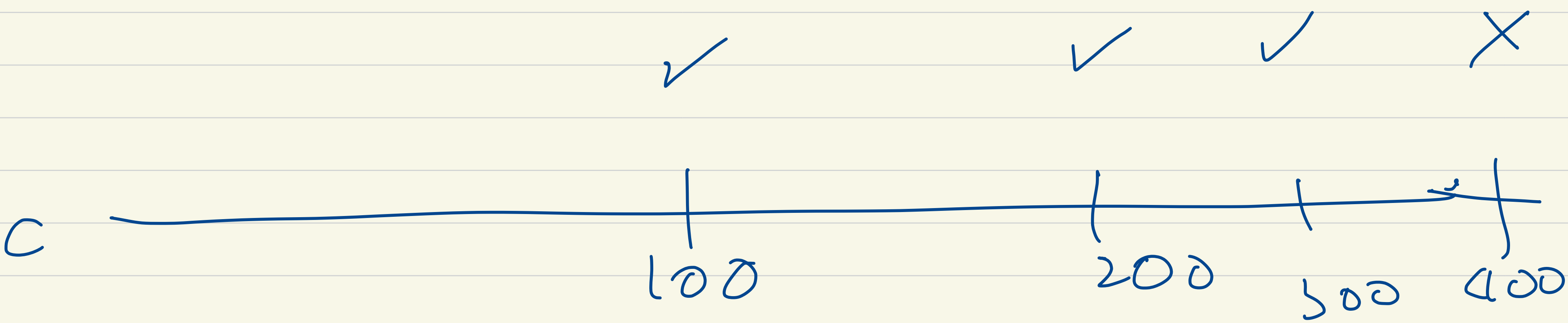
$$\boxed{\text{Find } x, y \quad \begin{cases} x + 2y \ge C \\ 2x + 9y \le 6 \\ 3x + 2y \le 7 \\ x, y \ge 0 \end{cases}}$$

$\Downarrow$

Suppose we can solve FEASIBILITY(c) $\forall c$

then we can binary search on value of

c to find the optimum LP value.



We can focus on solving Feasibility LPs.

Find x, y satisfying
$$x + 2y \leq 10$$
$$2x + 3y \leq 17$$
$$3x + 4y \leq 20$$
$$x \geq 0 \quad y \geq 0$$