LECTURE 18

* SOLVING LP FEASIBILITY

DEFINITION: [ε-SEPARATING LINE / PLANE]

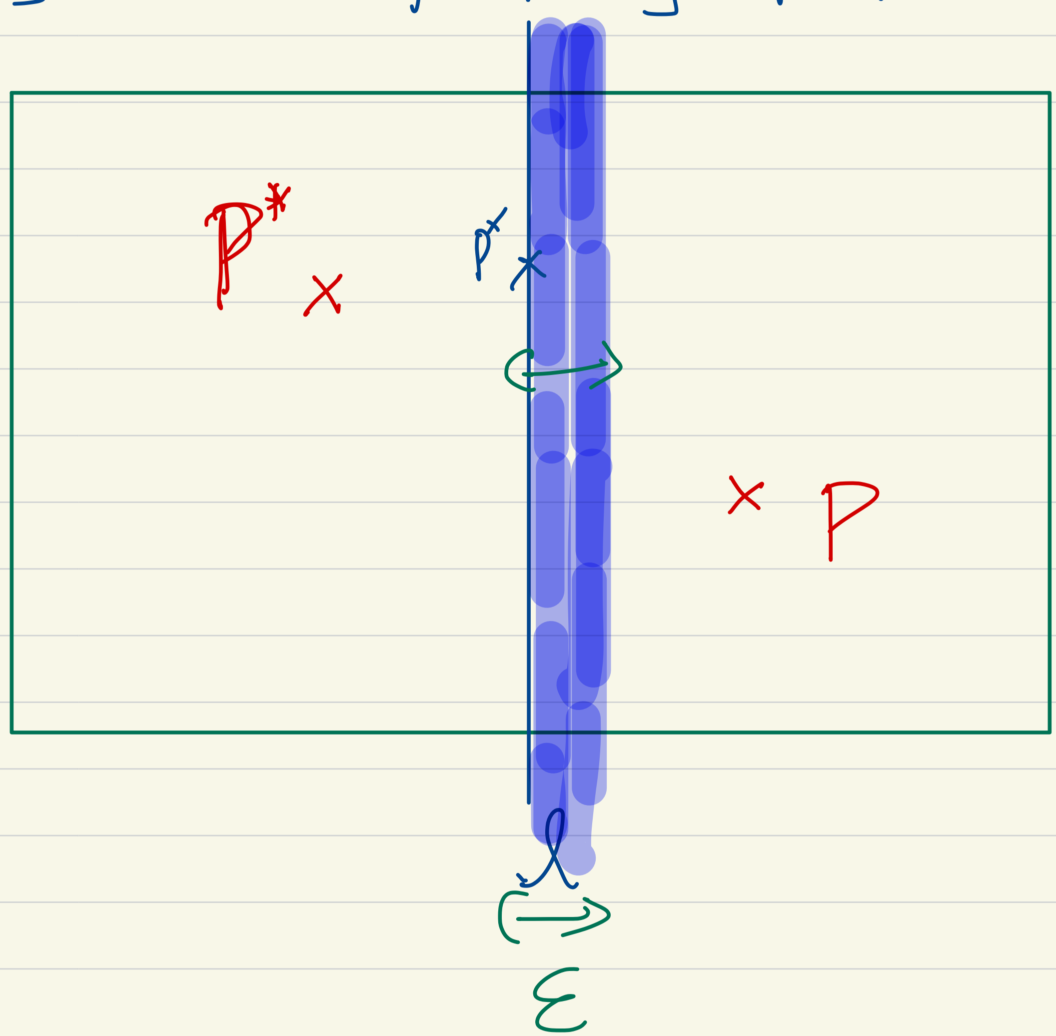
Given points p, p^*

An ε-separating line "separating p from p^*" is

1) p^* is on one side

2) p is on the other side

p is at least ε-away from the line

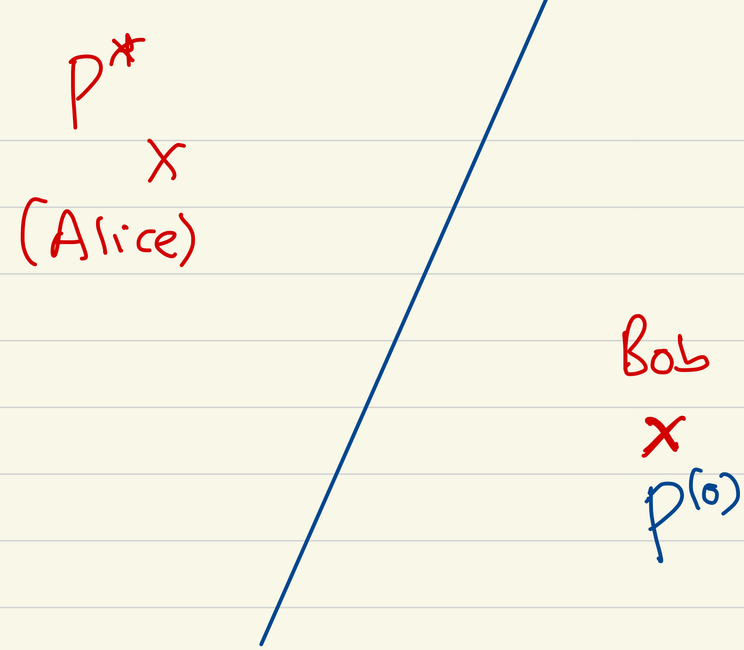


POINT PURSUIT GAME

SETUP: Alice is at point P^*

Bob is at $p^{(0)}$

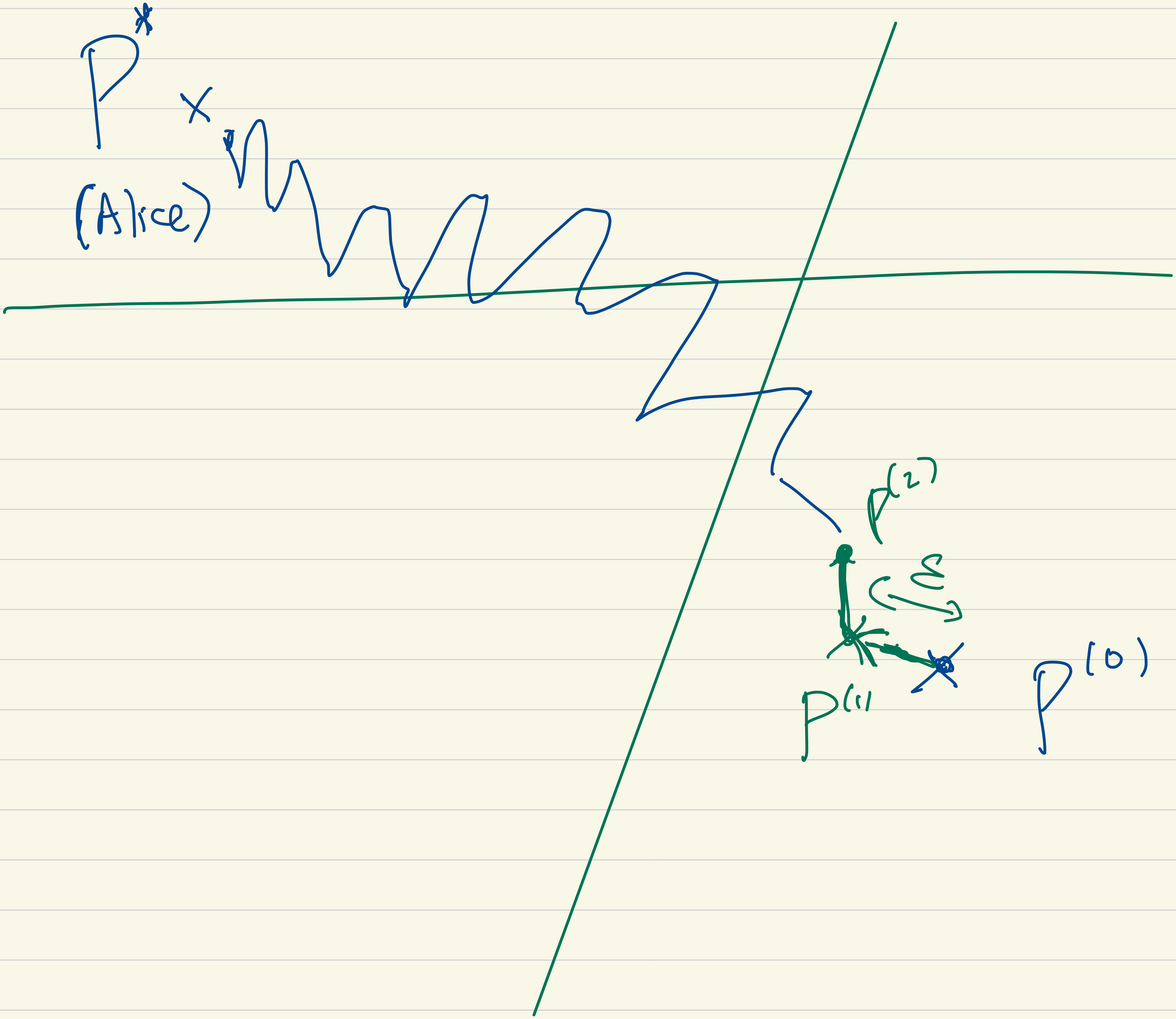
[Alice is giving directions for Bob to reach her]



At ROUND t:

1) Bob is at $p^{(t)}$

2) Alice announces a separating line between her location P^* AND Bob's current location $p^{(t)}$


P^*
(Alice)
$P^{(2)}$
ε
$P^{(1)}$
$P^{(0)}$

3) Bob updates his location

$$p^{(t)} \rightarrow p^{(t+1)}$$

Bob's STRATEGY:

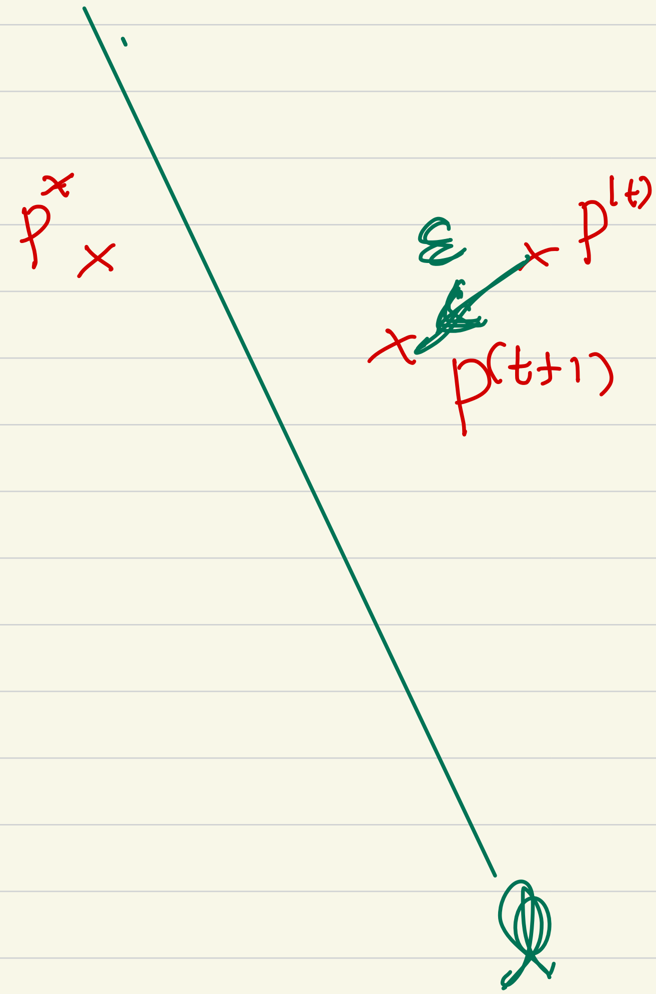
Move ε-distance directly towards the separating line

LINE: $ax + by = c$

Bob moves ε along the direction $\perp$ to line $\Rightarrow p^{t+1} = p^{(t)} + \varepsilon \cdot \vec{v}$

$$\vec{v} = (-b, a)$$

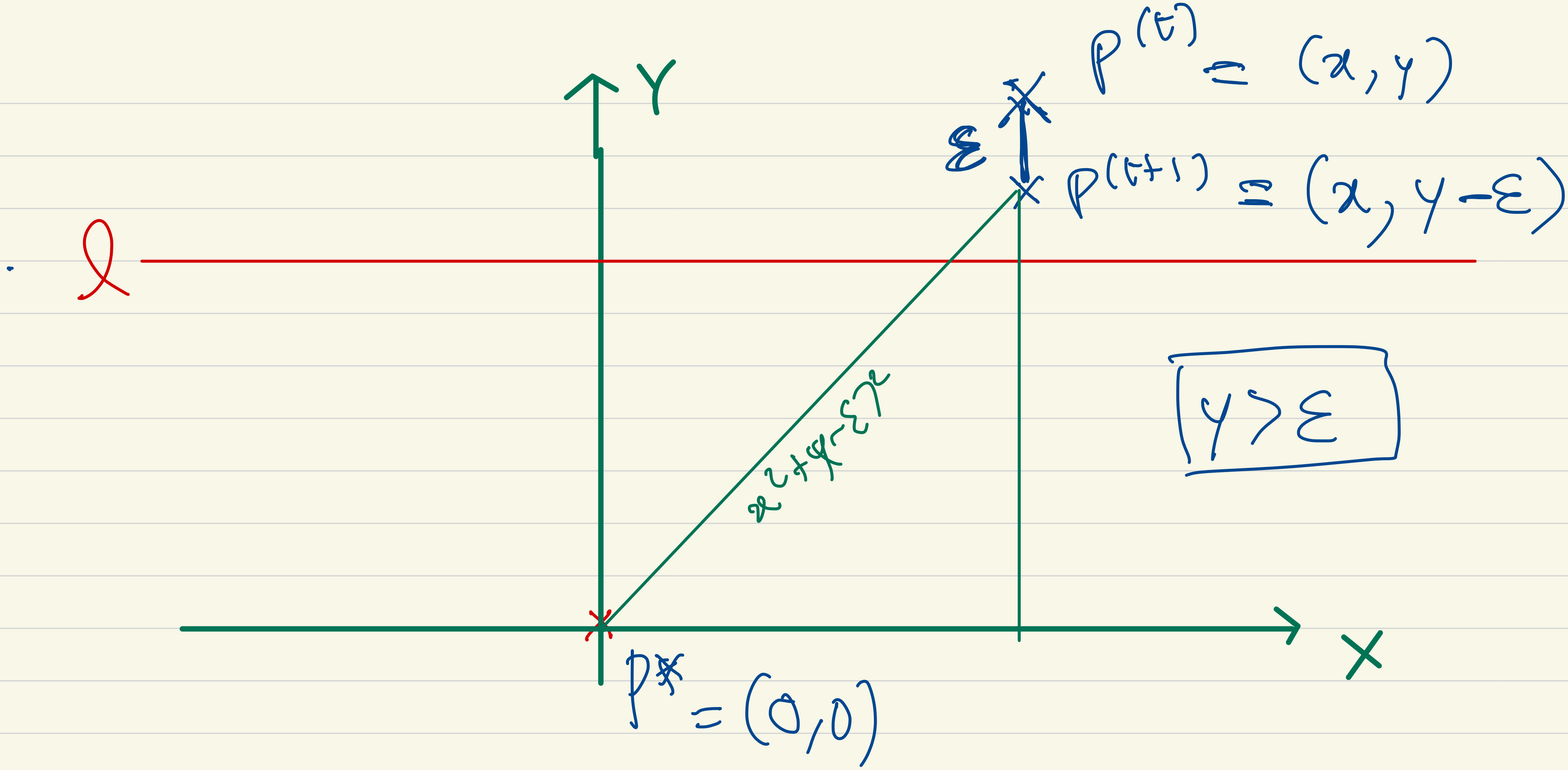
Claim: In each iteration, squared distance of Bob decreases by ε^2



$$\mathrm{dist}(p^{(t+1)}, p^*)^2 \le \mathrm{dist}(p^{(t)}, p^*)^2 - \varepsilon^2$$

By rotation & translation

→ Move p^* to origin



$$\text{dist}(p^t, p^*)^2 = x^2 + y^2$$

$$>$$

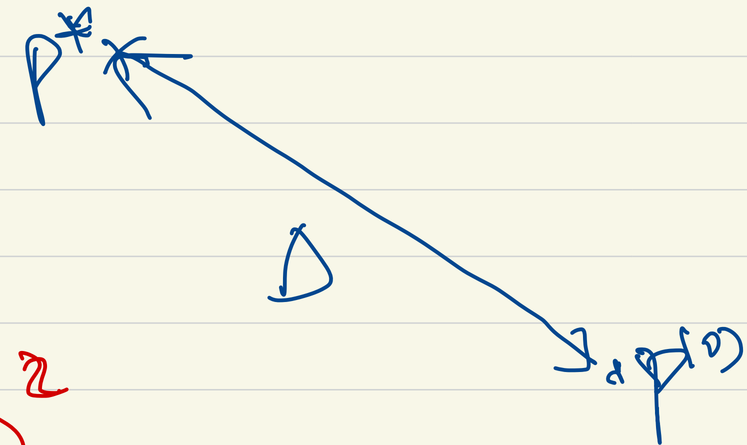
$$\text{dist}(p^{(t+1)}, p^*)^2 = x^2 + (y-\varepsilon)^2$$

difference $= 2y\varepsilon - \varepsilon^2$

$\geq 2\varepsilon^2 - \varepsilon^2 = \varepsilon^2$

$$\boxed{\text{We know } y \geq \varepsilon}$$

Theorem: If distance $(p^*, p^{(0)}) \leq D$ then game terminates in $O(D^2/\varepsilon^2)$ steps (irrespective of how Alice picks her separating lines)



Proof: $\text{dist}(p^*, p^{(0)})^2 = D^2$

By claim $\text{dist}(p^*, p^{(t)})$ decreases by ε^2 each step

$\Longrightarrow$ In D^2/ε^2 steps game has to terminate.

LP FEASIBILITY:

INPUT: SET OF LINEAR CONSTRAINTS

$$Ax \leq b,$$

GOAL: Find x satisfying all constraints

OR

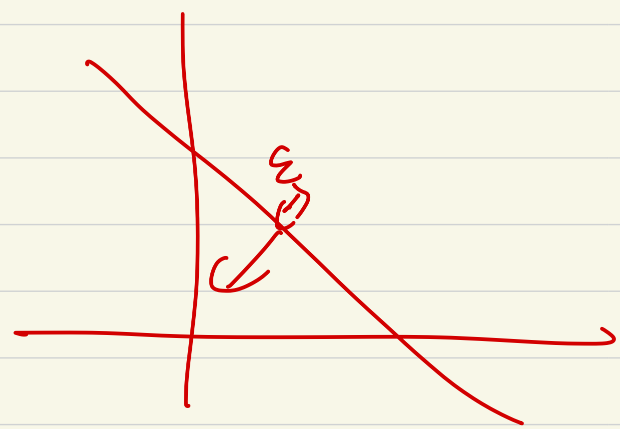
return NO POINT EXISTS.

GOAL*: Find x that is ε-close to satisfying all constraints

Example:

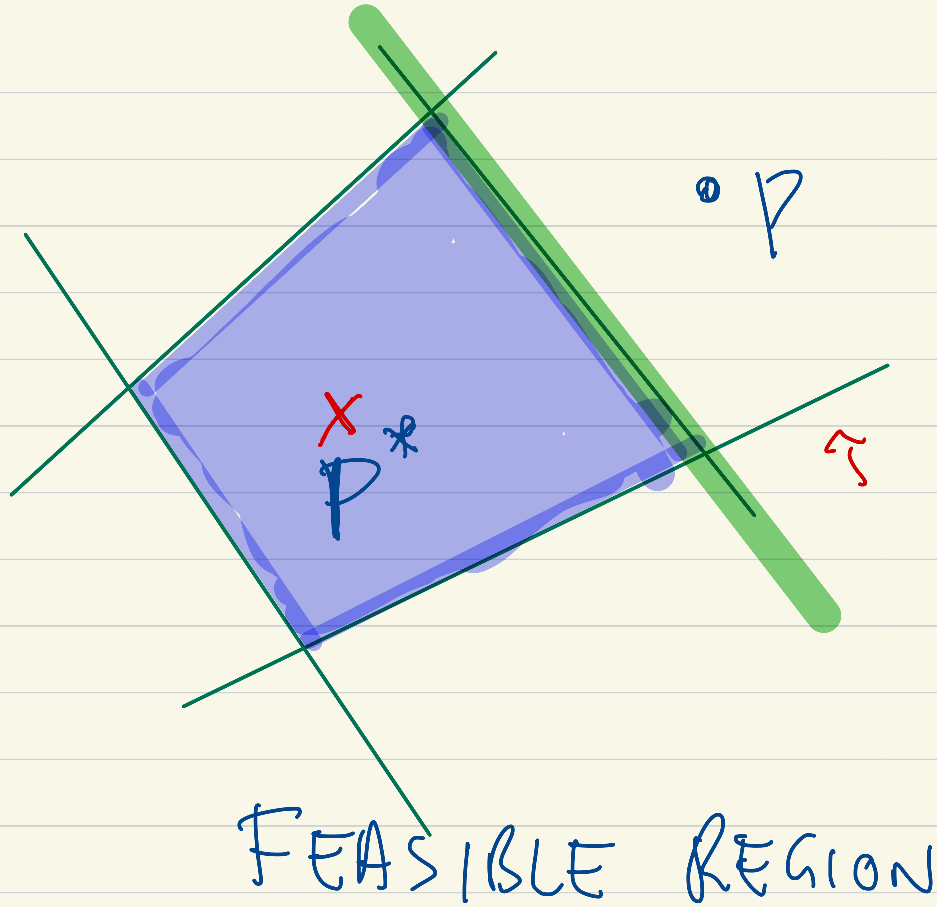
$$\left.\begin{aligned} x + y &\leq 10 \\ 2x + 3y &\leq 20 \\ 3x - 4y &\geq 5 \\ 8x + 9y &\geq 18 \\ &\vdots \end{aligned}\right\}$$

linear constraints



VIOLATED CONSTRAINTS

$\iff$ SEPARATING LINE/PLANE



If a point p violates a constraint ℓ

$\Updownarrow$

Line ℓ is a separating line between p and some feasible point p^*

ALGORITHM FOR LP FEASIBILITY:

→ SET $p^{(0)} \leftarrow (0,0)$

→ For $t = 1 \dots T$

DOES THE CURRENT POINT $p^{(t)}$: satisfy ALL CONSTRAINTS? within error ε

YES: RETURN $p^{(t)}$

NO: Let ℓ be a violated constraint. Move p^t directly towards ℓ

$p^{(t)} \longrightarrow p^{(t+1)}$

$x + y \leq 10$

$2x + 3y \leq 20$

$3x - 4y \geq 5$

$8x + 9y \geq 18$

$\vdots$

AFTER T iterations

RETURN "NO FEASIBLE SOLUTION WITHIN DISTANCE $\boxed{\varepsilon\sqrt{T}}$"

Alice Bob game : Start at distance D

$\Rightarrow$ game concludes in $\frac{D^2}{\varepsilon^2}$ steps

$$\boxed{\text{Set } T = D^2/\varepsilon^2 \Rightarrow D = \varepsilon\sqrt{T}}$$

ANALYSIS:

ALGO FOR CP FEASIBILITY

→ SET $P^{(0)} \leftarrow (0,0)$

→ For $t = 1 \ldots T$

1) DOES $P^{(t)}$ violate SOME CONSTRAINT by $> \varepsilon$?

NO: RETURN $P^{(t)}$

YES: Let ℓ be SOME SEPARATING line. Move towards line ℓ by distance ε.

RETURN "NO ε-FEASIBLE SOLN WITHIN

$x + y \leq 10$

$2x + 3y \leq 20$

$3x - 4y \geq 5$

$8x + 9y \geq 18$

$\vdots$

A CALL TO SEPARATION ORACLE!!

ε-SEPARATION $\boxed{\text{ORACLE}}$: $\Longleftrightarrow$ SUBROUTINE

An ε-separation oracle for an LP L is an algorithm

INPUT: A candidate solution x

OUTPUT: A constraint of LP L violated by $\geq \varepsilon$.

OR

x satisfies all constraints.

THEOREM: If LP L has a separation oracle $\mathcal{O}$

then we can find a ε-feasible point for L

with $\Theta\left(D^2/\varepsilon^2\right)$ calls to separation oracle

where

FAIR WORK ALLOCATION [EXPONENTIALLY MANY CONSTRAINTS]

INPUT: There are n workers

$\forall$ worker i, l_i = minimum work

u_i = maximum work

TOTAL WORK = W

(FAIR: No set of $n/4$ workers do more than $W/2$ work.)

GOAL: Assign work to all the workers while satisfying constraints

LP: x_i = work assigned to i^{th} worker

$$\sum x_i \geq W$$

$$l_i \leq x_i \leq u_i$$

$$\forall S \subseteq [n] \quad |S| = n/4$$

$$\sum_{i \in S} x_i \leq W/2$$

FAIRNESS

$$\binom{n}{n/4} \approx \exp(n)$$

SEPARATION ORACLE

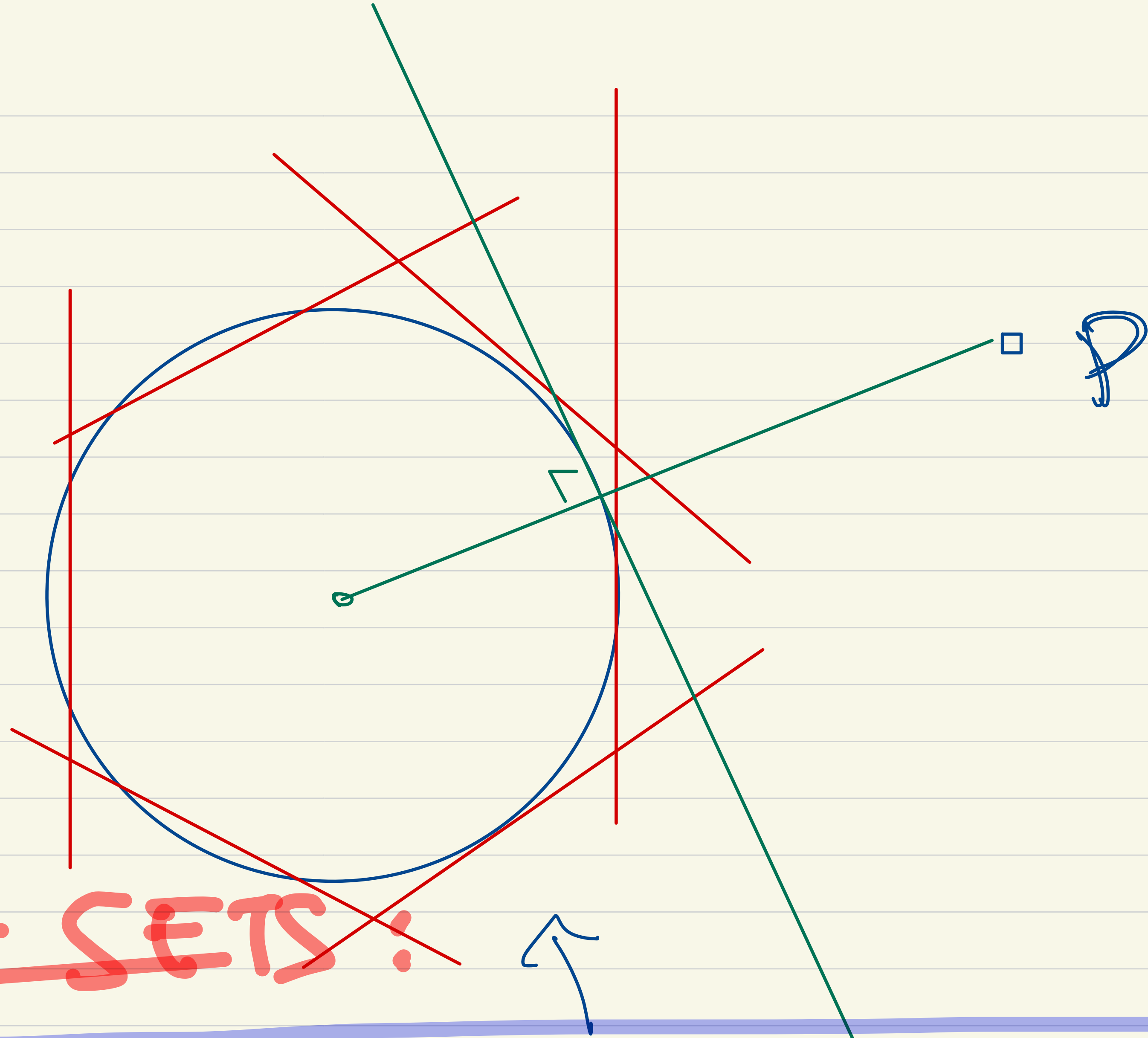
INPUT: $x_1 \dots x_n$ ← candidate solution

OUTPUT: Find a violated constraint.

1) Sort $x_1 \dots x_n$

2) Pick $S = \{$ largest $n/4$ values of $x_1 \dots x_n \}$

Check if $\sum_{i \in S} x_i > W/2$



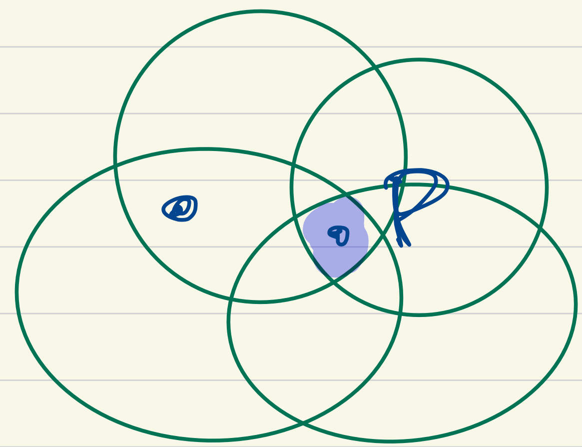
CONVEX SETS!

Sets defined by finitely / infinitely many linear constraints ??

INPUT: Circles $C_1 \ldots C_n$ on the plane

GOAL: Find a point inside all the circles.

Given any point P



1) Check every circle $C_1 \ldots C_n$

Find C_i s.t. $P \notin C_i$

Tangent to C_i gives separating line.